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The orthogonal eigenbasis and norms of eigenvectors in the spin Calogero–Sutherland model

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Abstract. Using a technique based on the Yangian Gelfand–Zetlin algebra and the associated Yangian Gelfand–Zetlin bases we construct an orthogonal basis of eigenvectors in the Calogero–Sutherland model with spin, and derive product-type formulae for norms of these eigenvectors.

1. Introduction

In this paper we study the spin generalization of the Calogero–Sutherland model which was proposed in [7] and later rediscovered in [4]. This model describes N particles with coordinates x_1, x_2, \dots, x_N moving along the circle of the unit radius ($0 \leq x_i \leq 2\pi$). Each particle carries a spin with n possible values, and the dynamics of the model is governed by the Hamiltonian

$$H_{\text{SCSM}} = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{1}{4} \sum_{1 \leq i \neq j \leq N} \frac{\beta(P_{i,j} + \beta)}{\sin^2\left(\frac{x_i - x_j}{2}\right)} \quad (1.1)$$

where β is a coupling constant and $P_{i,j}$ is the spin exchange operator for the particles i and j .

The scalar version of the H_{SCSM} ($n = 1$) has been studied over the course of the past 25 years starting with the work of Sutherland [26]. Among the recent advances one can point out the connection of the H_{SCSM} ($n = 1$) with the random matrix theory [11], exact computation of the dynamical correlation functions [14, 16, 20] and the intriguing connection with the Virasoro and the W -algebras [2]. To a large extent many of these developments, in particular the computation of the correlation functions, were based on the properties of the symmetric Jack polynomials which describe the orthogonal eigenbasis of the scalar Calogero–Sutherland model [25, 19].

Considerably less is known about the Calogero–Sutherland model with spin ($n \geq 2$). In the work of [4] the construction of eigenvectors for the Calogero–Sutherland model with general spin was proposed. This construction is based on the diagonalization of the Dunkl operators [10] by the non-symmetric Jack polynomials. Although the way to obtain the eigenvectors was pointed out in [4], the complete and orthogonal eigenbasis has not been constructed so far.

In the present paper we give a construction of such an eigenbasis in terms of the non-symmetric Jack polynomials and derive explicit product-type formulae for the norms of the eigenvectors.

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In the case of the scalar model the knowledge of explicit formulae for the norms of the Jack polynomials has been essential for the computation of the dynamical correlation functions. Therefore we believe that the results of our present work will turn out to be of use in the computation of the two-point dynamical correlation functions in the spin Calogero–Sutherland model (SCSM).

Let us now describe the main features of our construction. The principal role in it is played by the Yangian symmetry of the SCSM. As was discovered and emphasized in [4], the space of states in the model admits the action of the algebra $Y(\mathfrak{gl}_n)$ —the Yangian of \mathfrak{gl}_n [9, 23]. This action is given by the $n \times n$ operator-valued monodromy matrix $\|T_{a,b}(u)\|_{1 \leq a,b \leq n}$ which is regarded as the formal Taylor series in negative powers of the spectral parameter u . The centre of the Yangian action is generated by the operator coefficients $\Delta^{(s)}$ in the expansion of the quantum determinant $q \det T(u)$ of the monodromy matrix:

$$q \det T(u) = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{l(\sigma)} T_{1,\sigma(1)}(u) T_{2,\sigma(2)}(u-1) \cdots T_{n,\sigma(n)}(u-n+1) = \sum_{s=0}^{\infty} u^{-s} \Delta^{(s)} \quad (1.2)$$

$$[T_{a,b}(u), \Delta^{(s)}] = 0 \quad (a, b = 1, 2, \dots, n; s = 0, 1, 2, \dots). \quad (1.3)$$

The Hamiltonian of the model belongs to the Abelian algebra generated by the conserved charges $\Delta^{(s)}$ [4] and thereby commutes with the Yangian action.

In the scalar case ($n = 1$) the Yangian $Y(\mathfrak{gl}_1)$ coincides with its centre and is just the algebra of the conserved charges in the Calogero–Sutherland model. It is known [18, 19] that in this case the joint spectrum of the conserved charges is simple, and that the operators $\Delta^{(s)}$ are self-adjoint with respect to the scalar product relevant for the computation of quantities such as correlation functions. Hence the orthogonal eigenbasis of $H_{\text{SCSM}}(n = 1)$ is defined uniquely up to normalizations of eigenvectors as the eigenbasis of the Abelian algebra generated by the conserved charges $\Delta^{(s)}$.

In the situation when the spin is non-trivial ($n \geq 2$) the spectrum of the quantum determinant is not simple and thus the higher conserved charges alone are not sufficient to specify an orthogonal eigenbasis. To give such a specification, in this paper we use a maximal Abelian sub-algebra of $Y(\mathfrak{gl}_n)$ denoted by $A(\mathfrak{gl}_n)$ and known as the Yangian Gelfand–Zetlin algebra. This algebra includes the centre of the Yangian as a sub-algebra. The algebra $A(\mathfrak{gl}_n)$ was first studied by Cherednik [5] and subsequently by Nazarov and Tarasov [22, 23]. It is defined as the sub-algebra in $Y(\mathfrak{gl}_n)$ generated by all the centres in the chain of algebras

$$Y(\mathfrak{gl}_1) \subset Y(\mathfrak{gl}_2) \subset \cdots \subset Y(\mathfrak{gl}_n) \quad (1.4)$$

where $Y(\mathfrak{gl}_{m-1})$ is realized inside $Y(\mathfrak{gl}_m)$ as the sub-algebra generated by the entries of the sub-matrix $\|T_{a,b}(u)\|_{1 \leq a,b \leq m-1}$.

The generators of the Abelian algebra $A(\mathfrak{gl}_n)$ which appear in the SCSM possess the following two crucial properties:

- They are self-adjoint with respect to the relevant scalar product (defined in section 2).
- They are simultaneously diagonalizable and their joint spectrum is simple.

From these two properties it follows that, since the spin Calogero–Sutherland Hamiltonian belongs to the algebra $A(\mathfrak{gl}_n)$, the eigenbasis of the algebra $A(\mathfrak{gl}_n)$ is an orthogonal eigenbasis of the Hamiltonian.

Construction of this eigenbasis is the first main problem that we address in this paper. This construction is carried out in two steps. First, we describe the decomposition of the space of states in the model into irreducible sub-representations of the Yangian action and

point out the Yangian highest-weight vector in each of the irreducible components. These highest-weight vectors are expressed in terms of the non-symmetric Jack polynomials.

In [23] Nazarov and Tarasov gave construction of canonical bases, called Yangian Gelfand–Zetlin bases, for a wide class of Yangian representations which included all representations which appear as irreducible components of the Yangian action in the SCSM. The Yangian Gelfand–Zetlin base was first considered by Cherednik in [5] and is defined as the base where the action of the Abelian algebra $A(\mathfrak{gl}_n)$ is diagonal. It includes the highest-weight vector and ‘descendants’ which are obtained by acting on the highest-weight vector with appropriate creation operators described explicitly in [23].

Once we have found the irreducible Yangian decomposition of the space of states and have identified the highest-weight vectors, the results of [5, 23] can immediately be applied to describe the eigenbasis of $A(\mathfrak{gl}_n)$ within each of the irreducible sub-representations and hence in the entire space of states of the model.

The second main problem which we consider in this paper is computation of the norms of the eigenvectors. This computation is performed as follows. First, the norms of the Yangian highest-weight vectors are found by expressing them in terms of the norms of the non-symmetric Jack polynomials known from [8, 17, 24]. The norm formulae for the non-symmetric Jack polynomials are essential in this computation. The norm of the highest-weight vector is equal to the norm of a certain non-symmetric Jack polynomial multiplied by a non-trivial coefficient. The norms of the ‘descendants’ which constitute the rest of the Yangian Gelfand–Zetlin base are computed recursively from the norm of the highest-weight vector by using properties of the creation operators of [23].

The approach that we use in the present paper to construct the orthogonal eigenbasis and compute normalizations of the eigenvectors is by no means the only possible one. In this paper we consider the wavefunctions of the model as having both a coordinate and spin part so that a complete wavefunction is either totally symmetric or totally asymmetric. In this framework the Yangian symmetry and the associated Gelfand–Zetlin bases are the most natural to work with.

Another, equivalent, framework is provided by the polynomial presentation for the SCSM [15, 12, 3] which amounts to considering the Hamiltonian of the model as an operator which acts only on the coordinate part of the wavefunction. In this approach the spin part of the wavefunction does not appear explicitly but can always be recovered from the requirement that the complete wavefunction is to be totally symmetric or asymmetric. Polynomial eigenfunctions of the Hamiltonian are then obtained as certain linear combinations of non-symmetric Jack polynomials. These eigenfunctions exhibit rather complicated symmetry properties under permutation of variables and are called Jack polynomials with prescribed symmetry [3].

One can consider the problem of constructing an orthogonal eigenbasis of the model in terms of these polynomials and the problem of computing norms of the eigenvectors. The former problem was solved in [3]. The norms have not been computed so far in full generality (see however [12] for special cases and conjectures). It is clear that these norms are linear combinations of the known norms for the non-symmetric Jack polynomials [8, 17, 24], however, the problem of computing coefficients of these linear combinations seems to be technically rather complicated.

Now let us describe the contents of this paper. In section 2 we recall the definition of the SCSM. In section 3 the necessary background information on the Yangian and Yangian Gelfand–Zetlin bases is reviewed. The contents of this section largely follow the work of [23]. In section 4 we discuss properties of Yangian action in the SCSM. In section 5 the irreducible Yangian decomposition of the space of states is given. The main results in this

section are theorems 1 and 2. Section 6 contains derivation of the norm formulae for the Yangian highest-weight vectors. The main result here is proposition 12. In section 7 we give expressions for the ‘descendants’ of the highest-weight vectors. Proposition 14 gives formulae for their norms. The appendix contains proofs of some of the statements in the main text.

2. Definition of the model

In this section we will review the definition and a few known facts about the SCSM. In doing so we will closely follow the work of [4] where this model was introduced and extensively studied for the first time under the name of the dynamical model with long-range interaction. We would like to note, that the model which we define below is the gauge-transformed version of (1.1) [4].

2.1. The Hilbert space of states in the gauge-transformed SCSM

The space of states of the gauge-transformed SCSM [4] is a subspace in the tensor product

$$\mathcal{H} := \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}] \otimes (\otimes^N \mathbb{C}^n). \tag{2.1}$$

We fix the base $\{v_\epsilon\}_{\epsilon=1, \dots, n}$ in \mathbb{C}^n and define in $\otimes^N \mathbb{C}^n$ the Hermitian (sesquilinear) scalar product $\langle \cdot, \cdot \rangle_s$ by requiring pure tensors to be orthonormal:

$$\langle v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \dots \otimes v_{\epsilon_N}, v_{\tau_1} \otimes v_{\tau_2} \otimes \dots \otimes v_{\tau_N} \rangle_s := \prod_{i=1}^N \delta_{\epsilon_i, \tau_i} \quad (\epsilon_i, \tau_i = 1, 2, \dots, n). \tag{2.2}$$

In $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ we define the Hermitian scalar product $\langle \cdot, \cdot \rangle_c$ which depends on the parameter $\alpha \in \mathbb{R}_{>0}$. For $f(z_1, z_2, \dots, z_N), g(z_1, z_2, \dots, z_N) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ set

$$\begin{aligned} \langle f, g \rangle_c := & \frac{1}{N!} \left(\prod_{i=1}^N \oint_{|w_i|=1} \frac{dw_i}{2\pi \sqrt{-1} w_i} \right) \\ & \times \left(\prod_{i \neq j} \left(1 - \frac{w_i}{w_j} \right)^\alpha \overline{f(w_1, w_2, \dots, w_N)} g(w_1, w_2, \dots, w_N) \right) \end{aligned} \tag{2.3}$$

where the integration over each of the complex variables w_i is taken along the unit circle in the complex plane. The Hermitian scalar product $\langle \cdot, \cdot \rangle$ in the space \mathcal{H} is defined as the composition of the scalar products (2.2) and (2.3). For $f, g \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ and $u, v \in \otimes^N \mathbb{C}^n$ put

$$\langle f \otimes u, g \otimes v \rangle := \langle f, g \rangle_c \langle u, v \rangle_s \tag{2.4}$$

and extend the $\langle \cdot, \cdot \rangle$ on the entire space \mathcal{H} by requiring it to be sesquilinear.

The symmetric group \mathfrak{S}_N acts in the \mathcal{H} . For

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & N \\ \sigma(1) & \sigma(2) & \dots & \sigma(N) \end{pmatrix} \in \mathfrak{S}_N$$

there are two right actions K_σ and P_σ defined in the base $\{z_1^{m_1} z_2^{m_2} \dots z_N^{m_N} \otimes v_{\epsilon_1} \otimes v_{\epsilon_2} \otimes \dots \otimes v_{\epsilon_N}\}$ ($m_i \in \mathbb{Z}, 1 \leq \epsilon_i \leq n$) of the space \mathcal{H} by

$$\begin{Bmatrix} K_\sigma \\ P_\sigma \end{Bmatrix} z_1^{m_1} \dots z_N^{m_N} \otimes v_{\epsilon_1} \otimes \dots \otimes v_{\epsilon_N} = \begin{cases} z_1^{m_{\sigma(1)}} \dots z_N^{m_{\sigma(N)}} \otimes v_{\epsilon_1} \otimes \dots \otimes v_{\epsilon_N} \\ z_1^{m_1} \dots z_N^{m_N} \otimes v_{\epsilon_{\sigma(1)}} \otimes \dots \otimes v_{\epsilon_{\sigma(N)}}. \end{cases} \tag{2.5}$$

For the transposition $(i, j) \in \mathfrak{S}_N$ we will use the notations

$$K_{(i,j)} \equiv K_{i,j} \quad \text{and} \quad P_{(i,j)} \equiv P_{i,j}. \tag{2.6}$$

The operators $K_{i,j}$ and $P_{i,j}$ are easily seen to be self-adjoint and unitary with respect to the scalar product (2.4).

The SCSM can be defined in two versions—fermionic and bosonic [4]. Throughout this paper we will distinguish these versions by the sign of the parameter κ setting $\kappa = -$ (resp. $\kappa = +$) for the fermionic (resp. bosonic) case. The space of states $\mathcal{H}^{(\kappa)}$ in the gauge-transformed SCSM is then defined as follows:

$$\mathcal{H}^{(\kappa)} := \bigcap_{i=1}^{N-1} \text{Ker}(K_{i,i+1} P_{i,i+1} - \kappa 1) \subset \mathcal{H}. \tag{2.7}$$

Or, equivalently, as the image of the total asymmetrization or symmetrization operator:

$$\mathfrak{A}_N^{(\kappa)} := \sum_{\sigma \in \mathfrak{S}_N} (\kappa)^{l(\sigma)} K_\sigma P_\sigma. \tag{2.8}$$

The subspace $\mathcal{H}^{(\kappa)}$ inherits the scalar product (2.4) from the space \mathcal{H} . We will use the notation $\langle \cdot, \cdot \rangle_{(\kappa)}$ for this scalar product.

2.2. The Hamiltonian of the SCSM

The gauge-transformed SCSM Hamiltonian is defined through the Cherednik–Dunkl operators [10, 6, 4]:

$$d_i := \alpha z_i \frac{\partial}{\partial z_i} - i + \sum_{i < j} \frac{z_j}{z_j - z_i} (K_{i,j} - 1) - \sum_{i > j} \frac{z_i}{z_i - z_j} (K_{i,j} - 1) \quad (i = 1, 2, \dots, N) \tag{2.9}$$

which satisfy the relations of the degenerate affine Hecke algebra:

$$K_{i,i+1} d_i - d_{i+1} K_{i,i+1} = 1 \tag{2.10}$$

$$[d_j, K_{i,i+1}] = 0 \quad (j \neq i, i + 1) \tag{2.11}$$

$$[d_i, d_j] = 0. \tag{2.12}$$

We will consider the Cherednik–Dunkl operators as acting either in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ or in the first factor in $\mathcal{H} = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}] \otimes (\otimes^N \mathbb{C}^n)$ by expressions (2.9) and trivially in the second factor: $(\otimes^N \mathbb{C}^n)$ without always giving exact specification since this is unlikely to cause any confusion.

Relations (2.10)–(2.12) imply, in particular, that symmetric polynomials in d_1, d_2, \dots, d_N leave the subspaces $\mathcal{H}^{(\pm)}$ invariant [4]. In terms of the Cherednik–Dunkl operators the gauge-transformed Hamiltonian $H^{(\kappa)} \in \text{End}(\mathcal{H}^{(\kappa)})$ of the SCSM is

$$\begin{aligned} H^{(\kappa)} := & \sum_{i=1}^N \left\{ \left(\alpha z_i \frac{\partial}{\partial z_i} \right)^2 + (2i - N - 1) \alpha z_i \frac{\partial}{\partial z_i} \right\} \\ & + 2\alpha \sum_{i < j} \left\{ \frac{z_i}{z_i - z_j} \left(z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right) + \frac{z_i z_j}{(z_i - z_j)(z_j - z_i)} (-\kappa P_{i,j} + 1) \right\} \\ & + \frac{1}{12} N(N^2 - 1) = \sum_{i=1}^N \left(d_i - \frac{N+1}{2} \right)^2 - N(N+1)^2. \end{aligned} \tag{2.13}$$

Here to show the equality one has to use the relation

$$K_{i,j}f = \kappa P_{i,j}f \tag{2.14}$$

which holds for any $f \in \mathcal{H}^{(\kappa)}$ due to definition (2.7).

By a straightforward calculation one checks that the Cherednik–Dunkl operators are self-adjoint with respect to the scalar product (2.3) and hence the Hamiltonian $H^{(\kappa)}$ is self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_{(\kappa)}$. The physical Hamiltonian $H_{\text{SCSM}}^{(\kappa)}$ is obtained from the $H^{(\kappa)}$ by performing the gauge transformation [4]:

$$H_{\text{SCSM}}^{(\kappa)} = V^{\frac{1}{\alpha}} H^{(\kappa)} V^{-\frac{1}{\alpha}} = \sum_{i=1}^N \left(\alpha z_i \frac{\partial}{\partial z_i} \right)^2 + \sum_{i \neq j} \frac{z_i z_j}{(z_i - z_j)(z_j - z_i)} (-\kappa \alpha P_{i,j} + 1) \tag{2.15}$$

where

$$V = \left(\prod_{i=1}^N z_i \right)^{\frac{1-N}{2}} \prod_{i < j} z_i - z_j. \tag{2.16}$$

The Hamiltonian $H_{\text{SCSM}}^{(\kappa)}$ is identified up to the overall factor α^2 with the Hamiltonian (1.1) where $\beta = -\kappa 1/\alpha$ and $z_i = \exp(\sqrt{-1}x_i)$. The $H_{\text{SCSM}}^{(\kappa)}$ is self-adjoint with respect to the physical scalar product which is obtained from (2.4) by formally putting $\alpha = \infty$.

3. Yangian $Y(\mathfrak{gl}_n)$ and the Yangian Gelfand–Zetlin bases

In this section we summarize properties of the Yangian $Y(\mathfrak{gl}_n)$ which are used in this paper. The main attention is given to the Gelfand–Zetlin algebra and the canonical Yangian Gelfand–Zetlin bases in certain irreducible Yangian modules. The contents of this section, with the exception of the lemma 1 can be found in the works of [22, 23].

3.1. The definition of the Yangian $Y(\mathfrak{gl}_n)$ and the Gelfand–Zetlin algebra

The Yangian $Y(\mathfrak{gl}_n)$ is a unital associative algebra generated by the elements 1 and $T_{a,b}^{(s)}$ where $a, b = 1, \dots, n$ and $s = 1, 2, \dots$ that are subject to the following relations:

$$[T_{a,b}^{(r)}, T_{c,d}^{(s+1)}] - [T_{a,b}^{(r+1)}, T_{c,d}^{(s)}] = T_{c,b}^{(r)} T_{a,d}^{(s)} - T_{c,b}^{(s)} T_{a,d}^{(r)} \quad (r, s = 0, 1, 2, \dots) \tag{3.1}$$

where $T_{a,b}^{(0)} := \delta_{a,b} 1$.

Introducing the formal Taylor series in u^{-1}

$$T_{a,b}(u) = \delta_{a,b} + T_{a,b}^{(1)} u^{-1} + T_{a,b}^{(2)} u^{-2} + \dots \tag{3.2}$$

define $\overset{k}{T}(u)$ ($k = 1, 2$) as follows:

$$\overset{k}{T}(u) = \sum_{a,b=1}^n E_{a,b}^{(k)} \otimes T_{a,b}(u) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]]. \tag{3.3}$$

Here $E_{a,b}^{(k)}$ are the standard matrix units that are acting in the k th tensor factor \mathbb{C}^n . If we put

$$R(u, v) = \text{id} + \frac{1}{u - v} \sum_{a,b=1}^n E_{a,b}^{(1)} \otimes E_{b,a}^{(2)} \tag{3.4}$$

then the defining relations of $Y(\mathfrak{gl}_n)$ are

$$R(u, v) \overset{1}{T}(u) \overset{2}{T}(v) = \overset{2}{T}(v) \overset{1}{T}(u) R(u, v). \tag{3.5}$$

Let $i = (i_1, \dots, i_m)$ and $j = (j_1, \dots, j_m)$ be two sequences of indices such that

$$1 \leq i_1 < \dots < i_m \leq n \quad \text{and} \quad 1 \leq j_1 < \dots < j_m \leq n. \tag{3.6}$$

Let \mathfrak{S}_m be the symmetric group of degree m . Define

$$Q_{ij}(u) = \sum_{\sigma \in \mathfrak{S}_m} (-1)^{l(\sigma)} T_{i_1, j_{\sigma(1)}}(u) T_{i_2, j_{\sigma(2)}}(u-1) \cdots T_{i_m, j_{\sigma(m)}}(u-m+1) \tag{3.7}$$

and

$$A_0(u) = 1 \quad A_m(u) = Q_{ii}(u) \quad (m = 1, \dots, n) \tag{3.8}$$

$$B_m(u) = Q_{ij}(u) \quad C_m(u) = Q_{ji}(u) \quad D_m(u) = Q_{jj}(u) \quad (m = 1, \dots, n-1) \tag{3.9}$$

where $i = (1, \dots, m)$ and $j = (1, \dots, m-1, m+1)$.

The following propositions can be found in [22].

Proposition 1 ([22]). (a) The coefficients of $A_n(u)$ belong to the centre of the algebra $Y(\mathfrak{gl}_n)$.
 (b) All the coefficients of $A_1(u), \dots, A_n(u)$ pairwise commute.

Proposition 2 ([22]). The following commutation relations hold in $Y(\mathfrak{gl}_n)$:

$$[A_m(u), B_l(v)] = 0 \quad \text{if } l \neq m \tag{3.10}$$

$$[C_m(u), B_l(v)] = 0 \quad \text{if } l \neq m \tag{3.11}$$

$$[B_m(u), B_l(v)] = 0 \quad \text{if } |l - m| \neq 1 \tag{3.12}$$

$$(u - v)[A_m(u), B_m(v)] = B_m(u)A_m(v) - B_m(v)A_m(u) \tag{3.13}$$

$$(u - v)[C_m(u), B_m(v)] = D_m(u)A_m(v) - D_m(v)A_m(u). \tag{3.14}$$

Proposition 3 ([22]). The following relation holds in $Y(\mathfrak{gl}_n)$:

$$C_m(u)B_m(u-1) = D_m(u)A_m(u-1) - A_{m+1}(u)A_{m-1}(u-1). \tag{3.15}$$

By relations (3.14), (3.15) we obtain

$$D_m(u)A_m(u+1) = A_{m+1}(u+1)A_{m-1}(u) - B_m(u)C_m(u+1). \tag{3.16}$$

By proposition 1, the coefficients $A_m^{(s)}$ of the series $A_1(u), \dots, A_n(u)$:

$$A_m(u) = \sum_{s \geq 0} u^{-s} A_m^{(s)} \quad (m = 1, 2, \dots, n) \tag{3.17}$$

generate the commutative sub-algebra in $Y(\mathfrak{gl}_n)$. This algebra is called Gelfand–Zetlin algebra and is denoted by $A(\mathfrak{gl}_n)$.

The following lemma will be used in the next section.

Lemma 1. Let $*$: $Y(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_n)$ be the algebra anti-involution such that

$$T_{a,b}^{(1)*} = T_{b,a}^{(1)} \quad T_{a,b}^{(2)*} = T_{b,a}^{(2)} \quad \text{and} \quad A_n^{(t)*} = A_n^{(t)} \quad (t = 0, 1, 2, \dots). \tag{3.18}$$

Then

$$T_{a,b}^{(s)*} = T_{b,a}^{(s)} \quad \text{for all } s = 0, 1, 2, \dots \tag{3.19}$$

Proof. The lemma is proven by induction in s . Suppose $T_{a,b}^{(r)*} = T_{b,a}^{(r)}$ hold for all $r \leq s$. Then the relations of the Yangian (3.1) and

$$T_{a,b}^{(1)*} = T_{b,a}^{(1)} \quad T_{a,b}^{(2)*} = T_{b,a}^{(2)} \tag{3.20}$$

entail

$$T_{a,b}^{(s+1)*} = T_{b,a}^{(s+1)} \quad (a \neq b) \quad \text{and} \quad (T_{a,a}^{(s+1)} - T_{b,b}^{(s+1)})^* = T_{a,a}^{(s+1)} - T_{b,b}^{(s+1)}. \tag{3.21}$$

And the condition on the quantum determinant:

$$A_n^{(t)*} = A_n^{(t)} \quad (t = 0, 1, 2, \dots) \tag{3.22}$$

gives

$$(T_{1,1}^{(s+1)} + T_{2,2}^{(s+1)} + \dots + T_{n,n}^{(s+1)})^* = T_{1,1}^{(s+1)} + T_{2,2}^{(s+1)} + \dots + T_{n,n}^{(s+1)}. \tag{3.23}$$

This completes the proof of the induction step. Taking conditions (3.20) as the induction base we obtain the statement of the lemma. \square

3.2. Yangian Gelfand–Zetlin bases

Let V be an irreducible finite-dimensional \mathfrak{gl}_n -module and $E_{a,b}$ be the generators of \mathfrak{gl}_n . Denote by v_λ the highest-weight vector in V :

$$E_{a,a}v_\lambda = \lambda_a v_\lambda \quad E_{a,b}v_\lambda = 0 \quad a < b. \tag{3.24}$$

Then each difference $\lambda_a - \lambda_{a+1}$ is a non-negative integer. We assume that each λ_a is also an integer. Denote by \mathcal{T}_λ the set of all arrays, Λ , with integral entries of the form

$$\begin{array}{ccccccc} \lambda_{n,1} & \lambda_{n,2} & \dots & \dots & \dots & \dots & \lambda_{n,n} \\ & \lambda_{n-1,1} & \dots & \dots & \dots & \dots & \lambda_{n-1,n-1} \\ & & \ddots & & & & \\ & & & \lambda_{2,1} & \lambda_{2,2} & & \\ & & & & \lambda_{1,1} & & \end{array} \tag{3.25}$$

where $\lambda_{n,i} = \lambda_i$ and $\lambda_i \geq \lambda_{m,i}$ for all i and m . The array, Λ , is called a Gelfand–Zetlin scheme if

$$\lambda_{m,i} \geq \lambda_{m-1,i} \geq \lambda_{m,i+1} \tag{3.26}$$

for all possible m and i . Denote by \mathcal{S}_λ the subset in \mathcal{T}_λ consisting of the Gelfand–Zetlin schemes.

There is a canonical decomposition of the space V into the direct sum of one-dimensional subspaces associated with the chain of sub-algebras

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_n. \tag{3.27}$$

These subspaces are parametrized by the elements $\Lambda \in \mathcal{S}_\lambda$. The subspace $V_\Lambda \subset V$ corresponding to $\Lambda \in \mathcal{S}_\lambda$ is contained in an irreducible \mathfrak{gl}_m -submodule of the highest weight $(\lambda_{m,1}, \lambda_{m,2}, \dots, \lambda_{m,m})$ for each $m = n - 1, n - 2, \dots, 1$. These conditions define V_Λ uniquely. [13].

Let us recall some facts about representations of the Yangian $Y(\mathfrak{gl}_n)$.

If we set $u' = u + h, v' = v + h (h \in \mathbb{C})$, relations (3.5) are also satisfied for (u', v') . Thus the map

$$T_{a,b}(u) \mapsto T_{a,b}(u + h) \tag{3.28}$$

defines an automorphism of the algebra $Y(\mathfrak{gl}_n)$. So if there is a representation, V , of $Y(\mathfrak{gl}_n)$, we can construct another representation of $Y(\mathfrak{gl}_n)$ by the pullback through this automorphism.

We can regard the representation of the Lie algebra \mathfrak{gl}_n as the representation of $Y(\mathfrak{gl}_n)$. This transpires due to the existence of the homomorphism π_n from $Y(\mathfrak{gl}_n)$ to $U(\mathfrak{gl}_n)$: the universal enveloping algebra of \mathfrak{gl}_n :

$$\pi_n : T_{a,b}(u) \mapsto \delta_{a,b} + E_{b,a}u^{-1}. \tag{3.29}$$

Let V_λ be the irreducible \mathfrak{gl}_n -module whose highest weight is $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. We denote by $V_\lambda(h)$ the $Y(\mathfrak{gl}_n)$ -module obtained from V_λ by the pullback through this homomorphism and the automorphism (3.28).

The Yangian $Y(\mathfrak{gl}_n)$ has the coproduct $\Delta : Y(\mathfrak{gl}_n) \rightarrow Y(\mathfrak{gl}_n) \otimes Y(\mathfrak{gl}_n)$. It is given as follows:

$$\Delta(T_{a,b}(u)) = \sum_{c=1}^n T_{a,c}(u) \otimes T_{c,b}(u). \tag{3.30}$$

So if there are representations $V_i (i = 1, \dots, M)$ of the Yangian $Y(\mathfrak{gl}_n)$, we can construct the representation $V_1 \otimes V_2 \otimes \dots \otimes V_M$ of $Y(\mathfrak{gl}_n)$:

$$\begin{aligned} T_{a,b}(u)(v_1 \otimes v_2 \otimes \dots \otimes v_M) &= \Delta^{(n-1)} \circ \dots \circ \Delta^{(2)}(T_{a,b}(u))(v_1 \otimes v_2 \otimes \dots \otimes v_M) \\ &= \sum_{k_1 \dots k_{M-1}} T_{a,k_1}(u)v_1 \otimes T_{k_1,k_2}(u)v_2 \otimes \dots \otimes T_{k_{M-1},b}(u)v_M. \end{aligned} \tag{3.31}$$

From now on we consider the following representation of the Yangian $Y(\mathfrak{gl}_n)$:

$$W = V_{\lambda^{(1)}}(h^{(1)}) \otimes V_{\lambda^{(2)}}(h^{(2)}) \otimes \dots \otimes V_{\lambda^{(M)}}(h^{(M)}) \tag{3.32}$$

where we assume that $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$.

Let us set $\rho_0(u) = 1$ and for $m = 1, \dots, n$ let us define

$$\rho_m(u) = \prod_{s=1}^M \prod_{i=1}^m (u - i + 1 + h^{(s)}) \tag{3.33}$$

and

$$a_m(u) = \rho_m(u)A_m(u) \quad m = 0, \dots, n \tag{3.34}$$

$$b_m(u) = \rho_m(u)B_m(u) \quad m = 1, \dots, n - 1 \tag{3.35}$$

$$c_m(u) = \rho_m(u)C_m(u) \quad m = 1, \dots, n - 1 \tag{3.36}$$

$$d_m(u) = \rho_m(u)D_m(u) \quad m = 1, \dots, n - 1. \tag{3.37}$$

Then $a_m(u)$, $b_m(u)$, $c_m(u)$ and $d_m(u)$ are polynomials in u , and due to proposition 2 and (3.16), they satisfy

$$[a_m(u), b_l(v)] = 0 \quad \text{if } l \neq m \tag{3.38}$$

$$[c_m(u), b_l(v)] = 0 \quad \text{if } l \neq m \tag{3.39}$$

$$[b_m(u), b_l(v)] = 0 \quad \text{if } |l - m| \neq 1 \tag{3.40}$$

$$(u - v)[a_m(u), b_m(v)] = b_m(u)a_m(v) - b_m(v)a_m(u) \tag{3.41}$$

$$(u - v)[c_m(u), b_m(v)] = d_m(u)a_m(v) - d_m(v)a_m(u) \tag{3.42}$$

$$d_m(u)a_m(u + 1) = a_{m+1}(u + 1)a_{m-1}(u) - b_m(u)c_m(u + 1). \tag{3.43}$$

Let us fix a set of Gelfand–Zetlin schemes

$$\Lambda^{(s)} = (\lambda_{m,i} | 1 \leq i \leq m \leq n) \in \mathcal{T}_{\lambda^{(s)}} (s = 1, \dots, M) \tag{3.44}$$

and define the following polynomials for $m = 0, \dots, n$.

$$\varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(M)}}(u) = \prod_{s=1}^M \prod_{i=1}^m (u + \lambda_{m,i}^{(s)} - i + 1 + h^{(s)}). \tag{3.45}$$

Note that all the zeros of the m th polynomial

$$v_{m,i}^{(s)} = i - \lambda_{m,i}^{(s)} - 1 - h^{(s)} \tag{3.46}$$

are pairwise distinct due to our assumption on the parameters $h^{(1)}, \dots, h^{(M)}$.

For the pairs (m, m') ($1 \leq m' \leq m \leq n$), we introduce the ordering,

$$(m, m') < (l, l') \Leftrightarrow m' < l' \quad \text{or} \quad (m' = l' \text{ and } m > l). \tag{3.47}$$

Let $v_{\text{hvw}} \in W$ be the vector, which is the tensor product of the highest-weight vectors $v_{\text{hvw}}^{(s)}$ of the Lie algebra \mathfrak{gl}_n ($s = 1, \dots, M$). Then consider the following vector in W

$$v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}} = \prod_{(m, m')}^{\rightarrow} \left(\prod_{\substack{(s,t) \\ 1 \leq t \leq \lambda_{n,m'}^{(s)} - \lambda_{m,m'}^{(s)}}} b_m(v_{m,m'}^{(s)} - t) \right) v_{\text{hvw}}. \tag{3.48}$$

Here for each fixed m the elements $b_m(v_{m,m'}^{(s)} - t) \in \text{End}(W)$ commute because of relation (3.40).

Then the following propositions are satisfied (see [23]).

Proposition 4 ([23]). For every $m = 1, \dots, n$ we have the equality

$$a_m(u) v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}} = \varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(M)}}(u) v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}. \tag{3.49}$$

Proposition 5 ([23]). If $\Lambda^{(r)} \notin \mathcal{S}_{\lambda^{(r)}}$ for some $r \in \{1, \dots, M\}$, then $v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}} = 0$.

Proposition 6 ([23]). If $\Lambda^{(r)} \in \mathcal{S}_{\lambda^{(r)}}$ for every $r \in \{1, \dots, M\}$, then $v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}} \neq 0$.

Proposition 7 ([23]). $Y(\mathfrak{gl}_n)$ -module W is irreducible if $h^{(r)} - h^{(s)} \notin \mathbb{Z}$ for all $r \neq s$.

By propositions 4 and 6 and the fact that if $(\Lambda^{(1)}, \dots, \Lambda^{(M)}) \neq (\tilde{\Lambda}^{(1)}, \dots, \tilde{\Lambda}^{(M)})$ ($\forall r, \Lambda^{(r)}, \tilde{\Lambda}^{(r)} \in \mathcal{S}_{\lambda^{(r)}}$) then $\exists m$ st $\varpi_{m, \Lambda^{(1)}, \dots, \Lambda^{(M)}}(u) \neq \varpi_{m, \tilde{\Lambda}^{(1)}, \dots, \tilde{\Lambda}^{(M)}}(u)$, one can show the following.

Proposition 8. $v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}(\Lambda^{(r)} \in \mathcal{S}_{\lambda^{(r)}})$ for every $r \in \{1, \dots, M\}$ form a base of W .

4. Yangian in the spin Calogero–Sutherland model

In this section we recall the definition of the Yangian action in the SCSM [4] and establish some properties of this action—in particular the self-adjointness of the operators giving the action of the Gelfand–Zetlin algebra (proposition 10).

Following [4] for $\kappa = \pm$ define the monodromy operator $\hat{T}_0^{(\kappa)}(u) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathcal{H})[[u^{-1}]]$ by

$$\hat{T}_0^{(\kappa)}(u) = \sum_{a,b=1}^n E_{a,b} \otimes \hat{T}_{a,b}^{(\kappa)}(u) := \left(1 + \frac{P_{0,1}}{u - \kappa d_1}\right) \left(1 + \frac{P_{0,2}}{u - \kappa d_2}\right) \cdots \left(1 + \frac{P_{0,N}}{u - \kappa d_N}\right) \tag{4.1}$$

the $P_{0,i}$ in this definition is the permutation operator of the zeroth and i th tensor factors \mathbb{C}^n in the tensor product

$$\mathbb{C}_0^n \otimes \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}] \otimes \mathbb{C}_1^n \otimes \mathbb{C}_2^n \otimes \cdots \otimes \mathbb{C}_N^n = \mathbb{C}_0^n \otimes \mathcal{H}. \tag{4.2}$$

The $E_{a,b} \in \text{End}(\mathbb{C}^n)$ is the standard matrix unit in the basis $\{v_\epsilon\}$ introduced before definition (2.2). The operators $\hat{T}_{a,b}^{(\kappa),(s)} \in \text{End}(\mathcal{H})$ obtained by expanding the monodromy matrix $\hat{T}_{a,b}^{(\kappa)}(u)$:

$$\hat{T}_{a,b}^{(\kappa)}(u) = \delta_{a,b} 1 + \sum_{s \geq 1} u^{-s} \hat{T}_{a,b}^{(\kappa),(s)} \tag{4.3}$$

satisfy the defining relations (3.1) of the $Y(\mathfrak{gl}_n)$. By using the relations of the degenerate affine Hecke algebra (2.10)–(2.12) one can show [4] that the $\hat{T}_{a,b}^{(\kappa),(s)}$ leave the subspace $\mathcal{H}^{(\kappa)}$ invariant. We will set

$$T_{a,b}^{(\kappa)}(u) := \hat{T}_{a,b}^{(\kappa)}(u)|_{\mathcal{H}^{(\kappa)}} \in \text{End}(\mathcal{H}^{(\kappa)})[[u^{-1}]] \quad (a, b = 1, 2, \dots, n). \tag{4.4}$$

Denote the generating series which give the action of the Gelfand–Zetlin algebra in the Yangian representation defined by the monodromy matrix (4.4) by $A_1^{(\kappa)}(u), A_2^{(\kappa)}(u), \dots, A_n^{(\kappa)}(u)$. The $A_n^{(\kappa)}(u)$ is just the quantum determinant of the $T_{a,b}^{(\kappa)}(u)$. Hence

$$[A_n^{(\kappa)}(u), T_{a,b}^{(\kappa)}(v)] = 0 \quad (a, b = 1, 2, \dots, n). \tag{4.5}$$

The explicit expression for the quantum determinant [4]:

$$A_n^{(\kappa)}(u) = \prod_{i=1}^N \frac{u + 1 - \kappa d_i}{u - \kappa d_i} \tag{4.6}$$

shows that the SCSM Hamiltonian (2.13) is an element in the centre of the Yangian action and hence is an element in the Gelfand–Zetlin algebra.

Denote by O^\dagger the adjoint of an operator $O \in \text{End}(\mathcal{H}^{(\kappa)})$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{(\kappa)}$ defined in section 2. For $O(u) = \sum_{s \geq 0} u^{-s} O^{(s)} \in \text{End}(\mathcal{H}^{(\kappa)})[[u^{-1}]]$ we will write $O(u)^\dagger := \sum_{s \geq 0} u^{-s} O^{(s)\dagger}$.

Proposition 9.

$$T_{a,b}^{(\kappa)}(u)^\dagger = T_{b,a}^{(\kappa)}(u) \quad (\kappa = -, +). \tag{4.7}$$

Proof. By using lemma 1 to prove the proposition, it is sufficient to show that

$$T_{a,b}^{(\kappa),(1)\dagger} = T_{b,a}^{(\kappa),(1)} \quad T_{a,b}^{(\kappa),(2)\dagger} = T_{b,a}^{(\kappa),(2)} \tag{4.8}$$

and

$$A_n^{(\kappa)}(u)^\dagger = A_n^{(\kappa)}(u). \tag{4.9}$$

By using definition (4.4) and the same notation regarding the subscript 0 as in (4.1) we can write

$$T_0^{(\kappa),(1)} = \sum_{i=1}^N P_{0,i}$$

$$T_0^{(\kappa),(2)} = \left(\sum_{i=1}^N \kappa d_i P_{0,i} + \sum_{1 \leq i < j \leq N} P_{0,i} P_{0,j} \right) \Big|_{\mathcal{H}^{(\kappa)}} = \left(\sum_{i=1}^N \kappa d_i P_{0,i} + \sum_{1 \leq i < j \leq N} \kappa K_{i,j} P_{0,j} \right) \Big|_{\mathcal{H}^{(\kappa)}}.$$

The Cherednik–Dunkl operators d_i and the permutation operators $K_{i,j}$ ($i, j = 1, 2, \dots, N$) are self-adjoint with respect to the scalar product (2.3). On the other hand, for any $x, y \in \otimes^N \mathbb{C}^n$ we have

$$\langle P_{0,i} x, y \rangle_s = \langle x, P_{0,i}^t y \rangle_s \tag{4.10}$$

where superscript t_0 stands for the matrix transposition in the auxiliary space \mathbb{C}^n (4.1). Using the definitions of the scalar products (2.4) and $\langle \cdot, \cdot \rangle_{(\kappa)}$ we obtain (4.8).

Then (4.9) follows from the explicit expression for the quantum determinant (4.6) and the self-adjointness of the Cherednik–Dunkl operators with respect to the scalar product (2.4). \square

By using this proposition we can now establish the main result of this section.

Proposition 10.

$$A_m^{(\kappa)}(u)^\dagger = A_m^{(\kappa)}(u) \quad B_m^{(\kappa)}(u)^\dagger = C_m^{(\kappa)}(u) \quad C_m^{(\kappa)}(u)^\dagger = B_m^{(\kappa)}(u) \quad (\kappa = -, +). \tag{4.11}$$

Proof. Since in the following proof it is immaterial whether we are dealing with the fermionic or bosonic case, we will suppress the superscripts (κ) .

In [21], the proof of the following relations can be found:

$$\begin{aligned} T_{i_1, j_1}(u) T_{i_2, j_2}(u-1) \cdots T_{i_m, j_m}(u-m+1) E_{i_1, j_1}^{(1)} E_{i_2, j_2}^{(2)} \cdots E_{i_m, j_m}^{(m)} (H_m \otimes 1) \\ = (H_m \otimes 1) E_{i_1, j_1}^{(1)} E_{i_2, j_2}^{(2)} \cdots E_{i_m, j_m}^{(m)} T_{i_m, j_m}(u-m+1) \cdots T_{i_2, j_2}(u-1) T_{i_1, j_1}(u). \end{aligned} \tag{4.12}$$

Relations (4.12) are satisfied in $\text{End}(\mathbb{C}^n)^{\otimes m} \otimes Y(\mathfrak{gl}_n)[[u^{-1}]]$, and $H_m \in \text{End}(\mathbb{C}^n)^{\otimes m}$ is the asymmetrization map. By comparing the coefficient of $E_{i_1, j_1}^{(1)} E_{i_2, j_2}^{(2)} \cdots E_{i_m, j_m}^{(m)}$, we obtain

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_m} (-1)^{l(\sigma)} T_{i_1, j_{\sigma(1)}}(u) T_{i_2, j_{\sigma(2)}}(u-1) \cdots T_{i_m, j_{\sigma(m)}}(u-m+1) \\ = \sum_{\sigma \in \mathfrak{S}_m} (-1)^{l(\sigma)} T_{i_{\sigma(m)}, j_m}(u-m+1) \cdots T_{i_{\sigma(2)}, j_2}(u-1) T_{i_{\sigma(1)}, j_1}(u). \end{aligned} \tag{4.13}$$

Then if we take the adjoint, we have

$$\begin{aligned} \left(\sum_{\sigma \in \mathfrak{S}_m} (-1)^{l(\sigma)} T_{i_1, j_{\sigma(1)}}(u) T_{i_2, j_{\sigma(2)}}(u-1) \cdots T_{i_m, j_{\sigma(m)}}(u-m+1) \right)^\dagger \\ = \sum_{\sigma \in \mathfrak{S}_m} (-1)^{l(\sigma)} T_{j_1, i_{\sigma(1)}}(u) T_{j_2, i_{\sigma(2)}}(u-1) \cdots T_{j_m, i_{\sigma(m)}}(u-m+1). \end{aligned} \tag{4.14}$$

If we put $(i_1, \dots, i_m) = (1, \dots, m), (j_1, \dots, j_m) = (1, \dots, m)$, we obtain $A_m(u)^\dagger = A_m(u)$, and if we put $(i_1, \dots, i_m) = (1, \dots, m), (j_1, \dots, j_m) = (1, \dots, m-1, m+1)$ (resp. $(i_1, \dots, i_m) = (1, \dots, m-1, m+1), (j_1, \dots, j_m) = (1, \dots, m)$), we obtain $B_m(u)^\dagger = C_m(u)$ (resp. $C_m(u)^\dagger = B_m(u)$). \square

In section 7 we will see that the operator coefficients generated by $A_1^{(\kappa)}(u), \dots, A_n^{(\kappa)}(u)$ are simultaneously diagonalizable in $\mathcal{H}^{(\kappa)}$, and that their joint spectrum is multiplicity free. Since $A_1^{(\kappa)}(u), \dots, A_n^{(\kappa)}(u)$ are self-adjoint this implies that their common eigenvectors are mutually orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_{(\kappa)}$. Our main problem in this paper is to describe these eigenvectors and to compute their norms.

5. Decomposition of the space of states into irreducible Yangian submodules

In this section we construct the decomposition of the space of states of SCSM into irreducible submodules of the Yangian action. The procedure we follow is the one suggested in [4], it is based on the diagonalization of the Cherednik–Dunkl operators. The eigenvectors of the Cherednik–Dunkl operators, known as non-symmetric Jack polynomials, are reviewed

in the section 5.1. In section 5.2 we describe the decomposition of the space of states $\mathcal{H}^{(-)}$ in the fermionic model, the main result here is theorem 1. In section 5.3 we give the decomposition in the bosonic case.

5.1. Non-symmetric Jack polynomials

In this section we consider the Cherednik–Dunkl operators (2.9) as acting in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$. For $\alpha > 0$ the Cherednik–Dunkl operators are simultaneously diagonalizable. Their common eigenvectors form a base in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ and are sometimes called non-symmetric Jack polynomials. Here we will review some of the properties of these polynomials.

First we describe the labelling of the eigenvectors which will be convenient in the proofs of the statements we are going to make later. Let $\mathcal{M}_N := \{(m_1, m_2, \dots, m_N) \in \mathbb{Z}^N \mid m_1 \geq m_2 \geq \dots \geq m_N\}$ be the set of partitions which may have negative parts. There is a right action of the symmetric group \mathfrak{S}_N in \mathbb{Z}^N . For

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & N \\ \sigma(1) & \sigma(2) & \dots & \sigma(N) \end{pmatrix} \in \mathfrak{S}_N$$

and $(n_1, n_2, \dots, n_N) \in \mathbb{Z}^N$ it is defined by

$$\sigma(n_1, n_2, \dots, n_N) = (n_{\sigma(1)}, n_{\sigma(2)}, \dots, n_{\sigma(N)}). \tag{5.1}$$

For an $\mathbf{m} \in \mathcal{M}_N$ we define the subset $S^{\mathbf{m}}$ in \mathfrak{S}_N by

$$\sigma \in S^{\mathbf{m}} \tag{5.2}$$

iff for all $1 \leq i \leq N$ $\sigma(i) = \#\{j \leq i \mid m_{\sigma(j)} \geq m_{\sigma(i)}\} + \#\{j > i \mid m_{\sigma(j)} > m_{\sigma(i)}\}$. Let $\mathfrak{S}_N^{\mathbf{m}} \subset \mathfrak{S}_N$ be the subgroup leaving \mathbf{m} invariant. Then $S^{\mathbf{m}}$ intersects each of the right cosets of $\mathfrak{S}_N^{\mathbf{m}}$ in \mathfrak{S}_N at precisely one element, and the correspondence between $S^{\mathbf{m}}$ and the set of all distinct rearrangements of \mathbf{m} given by

$$\sigma \in S^{\mathbf{m}} \rightarrow \sigma \mathbf{m} = (m_{\sigma(1)}, m_{\sigma(2)}, \dots, m_{\sigma(N)}) \tag{5.3}$$

is bijective.

Some of the properties of the set $S^{\mathbf{m}}$ are summarized as follows:

if $\sigma \in S^{\mathbf{m}}$ then $\sigma(i, i + 1) \in S^{\mathbf{m}}$ iff $m_{\sigma(i)} \neq m_{\sigma(i+1)}$. (5.4)

if $\sigma \in S^{\mathbf{m}}$ then $l(\sigma) \left(:= \sum_{i < j} \theta(\sigma(i) > \sigma(j)) \right) = \sum_{i < j} \theta(m_{\sigma(i)} < m_{\sigma(j)})$ (5.5)

$\forall \sigma \in S^{\mathbf{m}}, \sigma \neq \text{id} \exists (i, i + 1)$ such that $m_{\sigma(i)} < m_{\sigma(i+1)}$ and $l(\sigma(i, i + 1)) = l(\sigma) - 1$. (5.6)

Here in the definition of the length $l(\sigma)$ we used the convention $\theta(x) = 1$ if x is true, $\theta(x) = 0$ if x is false.

In the set $S^{\mathbf{m}}$ we introduce the total ordering by setting

$$\sigma > \sigma' \tag{5.7}$$

iff the last non-zero element in $(m_{\sigma(1)} - m_{\sigma'(1)}, m_{\sigma(2)} - m_{\sigma'(2)}, \dots, m_{\sigma(N)} - m_{\sigma'(N)})$ is < 0 . Notice that the identity in \mathfrak{S}_N is the maximal element in $S^{\mathbf{m}}$ in this ordering. Then in the set of pairs $(\mathbf{m}, \sigma) (\mathbf{m} \in \mathcal{M}_N, \sigma \in S^{\mathbf{m}})$ the partial ordering is defined by

$$(\mathbf{m}, \sigma) > (\tilde{\mathbf{m}}, \tilde{\sigma}) \quad \text{iff} \quad \begin{cases} \mathbf{m} > \tilde{\mathbf{m}} & \text{or} \\ \mathbf{m} = \tilde{\mathbf{m}} & \sigma > \tilde{\sigma} \end{cases} \tag{5.8}$$

where $\mathbf{m} > \tilde{\mathbf{m}}$ means that \mathbf{m} is greater than $\tilde{\mathbf{m}}$ in the dominance (natural) ordering in \mathcal{M}_N [19].

The eigenvectors $\Phi_\sigma^{\mathbf{m}}(z) \in \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ of the Cherednik–Dunkl operators are labelled by the pairs (\mathbf{m}, σ) ($\mathbf{m} \in \mathcal{M}_N, \sigma \in S^{\mathbf{m}}$) and satisfy the following properties:

$$\Phi_\sigma^{\mathbf{m}}(z) = z_1^{m_{\sigma(1)}} z_2^{m_{\sigma(2)}} \dots z_N^{m_{\sigma(N)}} + \sum_{(\tilde{\mathbf{m}}, \tilde{\sigma}) < (\mathbf{m}, \sigma)} c_{(\mathbf{m}, \sigma); (\tilde{\mathbf{m}}, \tilde{\sigma})} z_1^{\tilde{m}_{\tilde{\sigma}(1)}} z_2^{\tilde{m}_{\tilde{\sigma}(2)}} \dots z_N^{\tilde{m}_{\tilde{\sigma}(N)}} \tag{5.9}$$

$$d_i \Phi_\sigma^{\mathbf{m}}(z) = \xi_i^{\mathbf{m}}(\sigma) \Phi_\sigma^{\mathbf{m}}(z) \quad \text{where } \xi_i^{\mathbf{m}}(\sigma) := \alpha m_{\sigma(i)} - \sigma(i) \quad (i = 1, 2, \dots, N) \tag{5.10}$$

$$K_{i,i+1} \Phi_\sigma^{\mathbf{m}}(z) = \mathcal{A}_i^{\mathbf{m}}(\sigma) \Phi_\sigma^{\mathbf{m}}(z) + \mathcal{B}_i^{\mathbf{m}}(\sigma) \Phi_{\sigma(i,i+1)}^{\mathbf{m}}(z) \tag{5.11}$$

where

$$\mathcal{A}_i^{\mathbf{m}}(\sigma) = \frac{1}{\xi_i^{\mathbf{m}}(\sigma) - \xi_{i+1}^{\mathbf{m}}(\sigma)}$$

$$\mathcal{B}_i^{\mathbf{m}}(\sigma) = \begin{cases} \frac{(\xi_i^{\mathbf{m}}(\sigma) - \xi_{i+1}^{\mathbf{m}}(\sigma))^2 - 1}{(\xi_i^{\mathbf{m}}(\sigma) - \xi_{i+1}^{\mathbf{m}}(\sigma))^2} & (m_{\sigma(i)} > m_{\sigma(i+1)}) \\ 0 & (m_{\sigma(i)} = m_{\sigma(i+1)}) \\ 1 & (m_{\sigma(i)} < m_{\sigma(i+1)}). \end{cases} \tag{5.12}$$

Notice that for $\sigma \in S^{\mathbf{m}}$ we have $\sigma(i + 1) = \sigma(i) + 1$ whenever $m_{\sigma(i)} = m_{\sigma(i+1)}$, and hence in this case (5.11) and (5.12) give

$$K_{i,i+1} \Phi_\sigma^{\mathbf{m}}(z) = \Phi_\sigma^{\mathbf{m}}(z) \quad (m_{\sigma(i)} = m_{\sigma(i+1)}). \tag{5.13}$$

For $\alpha > 0$, the set of N eigenvalues $(\xi_1^{\mathbf{m}}(\sigma), \xi_2^{\mathbf{m}}(\sigma), \dots, \xi_N^{\mathbf{m}}(\sigma))$ determines the pair (\mathbf{m}, σ) uniquely. Since the Cherednik–Dunkl operators are self-adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_c$ (2.3), this implies that the eigenvectors $\Phi_\sigma^{\mathbf{m}}(z)$ are mutually orthogonal:

$$\langle \Phi_\sigma^{\mathbf{m}}(z), \Phi_\kappa^{\mathbf{n}}(z) \rangle_c = \delta_{\mathbf{m}, \mathbf{n}} \delta_{\sigma, \kappa} \|\Phi_\sigma^{\mathbf{m}}(z)\|_c^2. \tag{5.14}$$

Their norms $\|\Phi_\sigma^{\mathbf{m}}(z)\|_c^2$ have been computed in [24] and for the q -deformed situation in [17, 8]. The product formulae for the norms $\|\Phi_{\text{id}}^{\mathbf{m}}(z)\|_c^2$ will be used in section 6 to derive product formulae for the norms of the Yangian highest-weight vectors.

5.2. Irreducible decomposition of the space of states with respect to the Yangian action.

Fermionic case

In this section we describe the decomposition of the space of states in the fermionic SCSM: $\mathcal{H}^{(-)}$ into irreducible subrepresentations with respect to the $Y(\mathfrak{gl}_n)$ -action (4.4) ($\kappa = -$).

Let $E^{\mathbf{m}} := \bigoplus_{\sigma \in S^{\mathbf{m}}} \mathbb{C} \Phi_\sigma^{\mathbf{m}}(z)$ ($\mathbf{m} \in \mathcal{M}_N$); and let

$$F^{\mathbf{m}} := (E^{\mathbf{m}} \otimes (\otimes^N \mathbb{C}^n)) \cap \mathcal{H}^{(-)}. \tag{5.15}$$

Then (5.10) implies that the space $F^{\mathbf{m}}$ is invariant with respect to the Yangian action defined by (4.4) with $\kappa = -$. And since the polynomials $\Phi_\sigma^{\mathbf{m}}(z)$ ($\mathbf{m} \in \mathcal{M}_N, \sigma \in S^{\mathbf{m}}$) form a base in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ we have

$$\mathcal{H}^{(-)} = \bigoplus_{\mathbf{m} \in \mathcal{M}_N} F^{\mathbf{m}}. \tag{5.16}$$

Expression (4.6) ($\kappa = -$) implies that, unless $F^{\mathbf{m}} = \emptyset$, $F^{\mathbf{m}}$ is an eigenspace of the quantum determinant with the eigenvalue

$$\prod_{i=1}^N \frac{u + 1 + \xi_i^{\mathbf{m}}(\text{id})}{u + \xi_i^{\mathbf{m}}(\text{id})} \tag{5.17}$$

and hence is an eigenspace of the Hamiltonian $H^{(-)}$ (2.13).

To describe each of the components F^m explicitly we need to make several definitions.

Let $W_{(-)}^m \subset \otimes^N \mathbb{C}^n$ ($m \in \mathcal{M}_N$) be defined by

$$W_{(-)}^m := \bigcap_{1 \leq i \leq N, \text{st} i = m_{i+1}} \text{Ker}(P_{i,i+1} + 1). \tag{5.18}$$

Note that $\dim W_{(-)}^m = 0$ unless $m \in \mathcal{M}_N^{(n)}$ where

$$\mathcal{M}_N^{(n)} := \{m \in \mathcal{M}_N \mid \#\{m_k \mid m_k = i\} \leq n (i \in \mathbb{Z})\}. \tag{5.19}$$

For $p \in \{1, 2, \dots, n\}$ let λ be the highest weight of the fundamental \mathfrak{gl}_n -module:

$$\lambda = (\underbrace{1, 1, \dots, 1}_p, \underbrace{0, 0, \dots, 0}_{n-p}) \quad (1 \leq p \leq n). \tag{5.20}$$

For a highest weight of this form and $h \in \mathbb{C}$ denote the corresponding $Y(\mathfrak{gl}_n)$ -module $V_\lambda(h)$ (see section 3.2) by $V_p(h)$. As a linear space the $V_p(h)$ is realized as the totally asymmetricized tensor product of \mathbb{C}^n :

$$V_p(h) = \bigcap_{i=1}^{p-1} \text{Ker}(P_{i,i+1} + 1) \subset \otimes^p \mathbb{C}^n \quad (1 \leq p \leq n) \tag{5.21}$$

with normalization chosen so that the \mathfrak{gl}_n highest-weight vector in $V_p(h)$ is

$$\omega_p := \sum_{\sigma \in \mathfrak{S}_p} (-1)^{l(\sigma)} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \dots \otimes v_{\sigma(p)}. \tag{5.22}$$

For an $m \in \mathcal{M}_N^{(n)}$ let M be the number of distinct elements in the sequence

$$m = (m_1, m_2, \dots, m_N).$$

And let p_s ($1 \leq p_s \leq n, s = 1, 2, \dots, M$) be the multiplicities of the elements in m :

$$\begin{aligned} m_1 = m_2 = \dots = m_{p_1} > m_{1+p_1} = m_{2+p_1} = \dots = m_{p_2+p_1} > \dots > m_{1+p_{M-1}+\dots+p_2+p_1} \\ = m_{2+p_{M-1}+\dots+p_2+p_1} = \dots = m_{p_M+\dots+p_2+p_1} \equiv N. \end{aligned} \tag{5.23}$$

With $\xi_i^m := \xi_i^m(\text{id})$ (5.10) set

$$h_m^{(s)} := \xi_{1+p_1+p_2+\dots+p_{s-1}}^m \quad (p_0 := 0, s = 1, 2, \dots, M). \tag{5.24}$$

Then for the linear space $W_{(-)}^m$ (5.18) we have

$$W_{(-)}^m = \begin{cases} V_{p_1}(h_m^{(1)}) \otimes V_{p_2}(h_m^{(2)}) \otimes \dots \otimes V_{p_M}(h_m^{(M)}) \subset \otimes^N \mathbb{C}^n & \text{when } m \in \mathcal{M}_N^{(n)} \\ \emptyset & \text{when } m \notin \mathcal{M}_N^{(n)}. \end{cases} \tag{5.25}$$

When $m \in \mathcal{M}_N^{(n)}$ the $W_{(-)}^m$ is the Yangian module with the Yangian action defined by coproduct (3.30).

For any $\sigma \in S^m$ (5.2) define $\check{\mathbb{R}}^{(-)}(\sigma) \in \text{End}(\otimes^N \mathbb{C}^n)$ by the following recursion relation

$$\check{\mathbb{R}}^{(-)}(\text{id}) := 1 \tag{5.26}$$

$$\check{\mathbb{R}}^{(-)}(\sigma(i, i+1)) := -\check{R}_{i,i+1}(\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma))\check{\mathbb{R}}^{(-)}(\sigma) \quad (m_{\sigma(i)} > m_{\sigma(i+1)}) \tag{5.27}$$

where the R -matrix is given by

$$\check{R}_{i,i+1}(u) := u^{-1} + P_{i,i+1}. \tag{5.28}$$

Due to the property (5.6) of the set S^m this recursion relation is sufficient to define $\check{\mathbb{R}}^{(-)}(\sigma)$ for all $\sigma \in S^m$. The definition of $\check{\mathbb{R}}^{(-)}(\sigma)$ is unambiguous by virtue of the Yang-Baxter equation satisfied by the R -matrix (5.28).

For $\mathbf{m} \in \mathcal{M}_N$ define the map $U_{(-)}^{\mathbf{m}} : \otimes^N \mathbb{C}^n \rightarrow \mathcal{H}$ by setting for $v \in \otimes^N \mathbb{C}^n$

$$U_{(-)}^{\mathbf{m}} v := \sum_{\sigma \in S^{\mathbf{m}}} \Phi_{\sigma}^{\mathbf{m}}(z) \otimes \check{\mathbb{R}}^{(-)}(\sigma)v. \tag{5.29}$$

Theorem 1. For any $\mathbf{m} \in \mathcal{M}_N$ we have

$$U_{(-)}^{\mathbf{m}} : W_{(-)}^{\mathbf{m}} \mapsto F^{\mathbf{m}}. \tag{5.30}$$

And $U_{(-)}^{\mathbf{m}}$ is an isomorphism of the $Y(\mathfrak{gl}_n)$ -modules $W_{(-)}^{\mathbf{m}}$ and $F^{\mathbf{m}}$.

The proof of this theorem is given in appendix A.

This theorem will allow us to use the results of [23] described in section 3 in order to construct in $F^{\mathbf{m}}$ the eigenbasis of the algebra $A(\mathfrak{gl}_n)$ generated by the coefficients of the series $A_1^{(-)}(u), A_2^{(-)}(u), \dots, A_n^{(-)}(u)$. For now let us notice that from this theorem it follows that the Yangian highest-weight vector $\Omega_{\mathbf{m}}^{(-)}$ in $F^{\mathbf{m}}$ is given by

$$\Omega_{\mathbf{m}}^{(-)} = U_{(-)}^{\mathbf{m}} \omega_{\mathbf{m}} = \sum_{\sigma \in S^{\mathbf{m}}} \Phi_{\sigma}^{\mathbf{m}}(z) \otimes \check{\mathbb{R}}^{(-)}(\sigma) \omega_{\mathbf{m}} \tag{5.31}$$

where the $\omega_{\mathbf{m}}$ is the highest-weight vector in $W_{(-)}^{\mathbf{m}}$:

$$\omega_{\mathbf{m}} := \omega_{p_1} \otimes \omega_{p_2} \otimes \dots \otimes \omega_{p_M}. \tag{5.32}$$

From corollary 3.9 in [23] it follows that the modules $F^{\mathbf{m}}$ are irreducible if $\alpha \notin \mathbb{Q}$ since in this case in (5.25) we have $h_m^{(s)} - h_m^{(r)} \notin \mathbb{Z}$ when $s \neq r$. By using results of [1] we can verify, that the $F^{\mathbf{m}}$ are irreducible under the weaker condition: $\alpha \in \mathbb{R} \setminus \mathbb{Q}_{\leq 0}$. The key statements of [1] which are used to come to this conclusion are:

- $V_{p_1}(h^{(1)}) \otimes V_{p_2}(h^{(2)})$ is irreducible iff the $Y(\mathfrak{gl}_n)$ -intertwiner $R_{12} : V_{p_1}(h^{(1)}) \otimes V_{p_2}(h^{(2)}) \rightarrow V_{p_2}(h^{(2)}) \otimes V_{p_1}(h^{(1)})$ and the inverse intertwiner R_{21} have no poles;
- $V_{p_1}(h^{(1)}) \otimes V_{p_2}(h^{(2)}) \otimes \dots \otimes V_{p_M}(h^{(M)})$ is irreducible iff $V_{p_r}(h^{(r)}) \otimes V_{p_s}(h^{(s)})$ is irreducible for all $1 \leq r < s \leq M$.

5.3. Irreducible decomposition of the space of states with respect to the Yangian action.

Bosonic case

The decomposition of the space of states of the bosonic SCSM: $\mathcal{H}^{(+)}$ into irreducible subrepresentations with respect to the $Y(\mathfrak{gl}_n)$ -action (4.4) ($\kappa = +$) is carried out along the same lines as the one for the fermionic case.

Let for $\mathbf{m} \in \mathcal{M}_N$ the $E^{\mathbf{m}}$ be defined as in the previous section; and let

$$B^{\mathbf{m}} := (E^{\mathbf{m}} \otimes (\otimes^N \mathbb{C}^n)) \cap \mathcal{H}^{(+)}. \tag{5.33}$$

Then (5.10) implies that the space $B^{\mathbf{m}}$ is invariant with respect to the Yangian action defined by (4.4) with $\kappa = +$; and since the polynomials $\Phi_{\sigma}^{\mathbf{m}}(z)$ ($\mathbf{m} \in \mathcal{M}_N, \sigma \in S^{\mathbf{m}}$) form a base in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$ we have

$$\mathcal{H}^{(+)} = \bigoplus_{\mathbf{m} \in \mathcal{M}_N} B^{\mathbf{m}}. \tag{5.34}$$

To describe each of the components $B^{\mathbf{m}}$ explicitly we make several definitions analogous to those made in the previous section.

Let $W_{(+)}^{\mathbf{m}} \subset \otimes^N \mathbb{C}^n$ ($\mathbf{m} \in \mathcal{M}_N$) be defined by

$$W_{(+)}^{\mathbf{m}} := \bigcap_{1 \leq i \leq N \text{ s.t. } m_i = m_{i+1}} \text{Ker}(P_{i,i+1} - 1). \tag{5.35}$$

For $p = 1, 2, \dots$ let λ be the following \mathfrak{gl}_n highest weight:

$$\lambda = (p, \underbrace{0, 0, \dots, 0}_{n-1}). \tag{5.36}$$

For a highest weight of this form and $h \in \mathbb{C}$ denote the corresponding $Y(\mathfrak{gl}_n)$ -module $V_\lambda(h)$ (see section 3.2) by $V^p(h)$. As a linear space the $V^p(h)$ is realized as the totally symmetrized tensor product of \mathbb{C}^n :

$$V^p(h) = \cap_{i=1}^{p-1} \text{Ker}(P_{i,i+1} - 1) \subset \otimes^p \mathbb{C}^n \quad (p = 1, 2, \dots). \tag{5.37}$$

We choose normalization so that the highest-weight vector in $V_p(h)$ is equal to $v_1^{\otimes p}$

As in the fermionic case, for an $\mathbf{m} \in \mathcal{M}_N$ let M be the number of distinct elements in the sequence $\mathbf{m} = (m_1, m_2, \dots, m_N)$. And let $p_s (s = 1, 2, \dots, M)$ be the multiplicities of the elements in the \mathbf{m} :

$$\begin{aligned} m_1 = m_2 = \dots = m_{p_1} > m_{1+p_1} = m_{2+p_1} = \dots = m_{p_2+p_1} > \dots > m_{1+p_{M-1}+\dots+p_2+p_1} \\ = m_{2+p_{M-1}+\dots+p_2+p_1} = \dots = m_{p_M+\dots+p_2+p_1} = N. \end{aligned} \tag{5.38}$$

With $\xi_i^{\mathbf{m}} := \xi_i^{\mathbf{m}}(\text{id})$ (5.10) set

$$h_m^{(s)} := -\xi_{1+p_1+p_2+\dots+p_{s-1}}^{\mathbf{m}} \quad (p_0 := 0, \quad s = 1, 2, \dots, M). \tag{5.39}$$

Then for the linear space $W_{(+)}^{\mathbf{m}}$ (5.35) we have

$$W_{(+)}^{\mathbf{m}} = V^{p_1}(h_m^{(1)}) \otimes V^{p_2}(h_m^{(2)}) \otimes \dots \otimes V^{p_M}(h_m^{(M)}) \subset \otimes^N \mathbb{C}^n \quad (\mathbf{m} \in \mathcal{M}_N^{(n)}). \tag{5.40}$$

The $W_{(+)}^{\mathbf{m}}$ is the Yangian module with the Yangian action defined by coproduct (3.30).

For any $\sigma \in S^{\mathbf{m}}$ (5.2) define $\check{\mathbb{R}}^{(+)}(\sigma) \in \text{End}(\otimes^N \mathbb{C}^n)$ by the following recursion relation

$$\check{\mathbb{R}}^{(+)}(\text{id}) := 1 \tag{5.41}$$

$$\check{\mathbb{R}}^{(+)}(\sigma(i, i + 1)) := \check{R}_{i,i+1}(-\xi_i^{\mathbf{m}}(\sigma) + \xi_{i+1}^{\mathbf{m}}(\sigma))\check{\mathbb{R}}^{(+)}(\sigma) \quad (m_{\sigma(i)} > m_{\sigma(i+1)}) \tag{5.42}$$

where the R -matrix $\check{R}_{i,i+1}(u)$ is given by (5.28).

As in the fermionic case, due to the property (5.6) of the set $S^{\mathbf{m}}$ this recursion relation is sufficient to define $\check{\mathbb{R}}^{(+)}(\sigma)$ for all $\sigma \in S^{\mathbf{m}}$. The definition of $\check{\mathbb{R}}^{(+)}(\sigma)$ is unambiguous by virtue of the Yang–Baxter equation satisfied by the R -matrix (5.28).

For $\mathbf{m} \in \mathcal{M}_N$ define the map $U_{(+)}^{\mathbf{m}} : \otimes^N \mathbb{C}^n \rightarrow \mathcal{H}$ by setting for $v \in \otimes^N \mathbb{C}^n$

$$U_{(+)}^{\mathbf{m}} v := \sum_{\sigma \in S^{\mathbf{m}}} \Phi_\sigma^{\mathbf{m}}(z) \otimes \check{\mathbb{R}}^{(+)}(\sigma)v. \tag{5.43}$$

Theorem 2. For any $\mathbf{m} \in \mathcal{M}_N$ we have

$$U_{(+)}^{\mathbf{m}} : W_{(+)}^{\mathbf{m}} \mapsto B^{\mathbf{m}}. \tag{5.44}$$

And $U_{(+)}^{\mathbf{m}}$ is an isomorphism of the $Y(\mathfrak{gl}_n)$ -modules $W_{(+)}^{\mathbf{m}}$ and $B^{\mathbf{m}}$.

We omit the proof of this theorem since it is a straightforward modification of the proof of the theorem 1 given in appendix A. From this theorem it follows that the Yangian highest-weight vector $\Omega_m^{(+)}$ in $B^{\mathbf{m}}$ is given by

$$\Omega_m^{(+)} = U_{(+)}^{\mathbf{m}} v_1^{\otimes N} = \sum_{\sigma \in S^{\mathbf{m}}} \Phi_\sigma^{\mathbf{m}}(z) \otimes \check{\mathbb{R}}^{(+)}(\sigma)v_1^{\otimes N}. \tag{5.45}$$

Theorem 2 will allow us to use the results of [23], summarized in section 3, in order to construct in $B^{\mathbf{m}}$ the eigenbasis of the algebra $A(\mathfrak{gl}_n)$ generated by the coefficients of the series $A_1^{(+)}(u), A_2^{(+)}(u), \dots, A_n^{(+)}(u)$.

6. Norms of the highest-weight vectors in the irreducible Yangian submodules

6.1. The fermionic case

In this section we will compute the norms $\langle \Omega_m^{(-)}, \Omega_m^{(-)} \rangle_{(-)}$ of the highest-weight vectors in each of the irreducible submodules F^m ($m \in \mathcal{M}_N^{(n)}$).

Let us fix an $m \in \mathcal{M}_N^{(n)}$. In this section and later on we will use the notations (5.23), (5.24). Let $\Phi^m(z) := \Phi_{\text{id}}^m(z)$. Consider the vector

$$\mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m) \tag{6.1}$$

where $\mathfrak{A}_N^{(-)}$ is the asymmetrization operator (2.8). Due to (5.11) and the definition of the space F^m (5.15) we have

$$\mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m) \in F^m \tag{6.2}$$

and comparing the \mathfrak{gl}_n -weights of the vector (6.1) with $\Omega_m^{(-)}$ we find that these vectors are proportional:

$$\mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m) = c(m)\Omega_m^{(-)} \quad (c(m) \in \mathbb{R}). \tag{6.3}$$

Now we observe that from the self-adjointness of the elementary permutations $K_{i,i+1}^\dagger = K_{i,i+1}$, $P_{i,i+1}^\dagger = P_{i,i+1}$ with respect to the scalar product (2.4) it follows that the asymmetrization operator is self-adjoint as well:

$$\mathfrak{A}_N^{(-)\dagger} = \mathfrak{A}_N^{(-)}. \tag{6.4}$$

Therefore we can write

$$\begin{aligned} \langle \mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m), \mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m) \rangle &= N! \langle \Phi^m(z) \otimes \omega_m, \mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m) \rangle \\ &= N! c(m) \langle \Phi^m(z) \otimes \omega_m, \Omega_m^{(-)} \rangle \end{aligned}$$

and by formula (5.31) and the orthogonality of the polynomials $\Phi_\sigma^m(z)$ with respect to the scalar product (2.3):

$$\langle \mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m), \mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m) \rangle = c(m)^2 \langle \Omega_m^{(-)}, \Omega_m^{(-)} \rangle_{(-)} \tag{6.5}$$

$$= N! c(m) \langle \omega_m, \omega_m \rangle_s \langle \Phi^m(z), \Phi^m(z) \rangle_c. \tag{6.6}$$

Using (5.32), (5.22) to compute the norm $\langle \omega_m, \omega_m \rangle_s$ we obtain:

$$\langle \Omega_m^{(-)}, \Omega_m^{(-)} \rangle_{(-)} = N! \left(\prod_{s=1}^M p_s \right) \frac{1}{c(m)} \langle \Phi^m(z), \Phi^m(z) \rangle_c. \tag{6.7}$$

The norms $\langle \Phi^m(z), \Phi^m(z) \rangle_c$ are known, and can be found in [24, 8, 17]. For completeness we will give a derivation of these norms later in this section. For now we will proceed to compute the coefficient $c(m)$.

Writing

$$\mathfrak{A}_N^{(-)}(\Phi^m(z) \otimes \omega_m) = \sum_{\sigma \in S^m} \Phi_\sigma^m(z) \otimes \psi_\sigma \quad (\psi_\sigma \in \otimes^N \mathbb{C}^n) \tag{6.8}$$

and (5.31)

$$\Omega_m^{(-)} = \sum_{\sigma \in S^m} \Phi_\sigma^m(z) \otimes \check{\mathbb{R}}^{(-)}(\sigma) \omega_m \tag{6.9}$$

from (6.3) we obtain

$$\psi_\sigma = c(m) \check{\mathbb{R}}^{(-)}(\sigma) \omega_m \quad (\sigma \in S^m). \tag{6.10}$$

Let $\bar{\sigma} \in S^m$ be the unique element of maximal length $l(\sigma)$ (5.5) in the set S^m . This element corresponds to the antidominant rearrangement of the parts in the partition \mathbf{m} :

$$m_{\bar{\sigma}(1)} \leq m_{\bar{\sigma}(2)} \leq \dots \leq m_{\bar{\sigma}(N)}. \tag{6.11}$$

We will find the coefficient $c(\mathbf{m})$ from (6.10) by comparing the vector $\psi_{\bar{\sigma}}$ with $\check{\mathbb{R}}^{(-)}(\bar{\sigma})\omega_{\mathbf{m}}$.

First we compute the $\psi_{\bar{\sigma}}$. Let $\mathfrak{S}_N^{\mathbf{m}} \subset \mathfrak{S}_N$ be the subgroup preserving the partition \mathbf{m} . Then

$$\mathfrak{A}_N^{(-)}(\Phi^{\mathbf{m}}(z) \otimes \omega_{\mathbf{m}}) = \sum_{\sigma \in S^{\mathbf{m}}} (-1)^{l(\sigma)} K_{\sigma} P_{\sigma} \sum_{\tau \in \mathfrak{S}_N^{\mathbf{m}}} (-1)^{l(\tau)} K_{\tau} P_{\tau} \Phi^{\mathbf{m}}(z) \otimes \omega_{\mathbf{m}} \tag{6.12}$$

and from (5.13), (5.32), (5.22)

$$\mathfrak{A}_N^{(-)}(\Phi^{\mathbf{m}}(z) \otimes \omega_{\mathbf{m}}) = \left(\prod_{s=1}^M p_s! \right) \sum_{\sigma \in S^{\mathbf{m}}} (-1)^{l(\sigma)} K_{\sigma} P_{\sigma} \Phi^{\mathbf{m}}(z) \otimes \omega_{\mathbf{m}}. \tag{6.13}$$

Lemma 2. For any element $\sigma \in S^m$ we have

$$K_{\sigma} \Phi^{\mathbf{m}}(z) = \kappa^{\mathbf{m}}(\sigma) \Phi_{\sigma}^{\mathbf{m}}(z) + \sum_{\sigma' \in S^{\mathbf{m}}: l(\sigma') < l(\sigma)} v^{\mathbf{m}}(\sigma, \sigma') \Phi_{\sigma'}^{\mathbf{m}}(z) \tag{6.14}$$

where $v^{\mathbf{m}}(\sigma, \sigma'), \kappa^{\mathbf{m}}(\sigma) \in \mathbb{R}$ and (5.12)

$$\kappa^{\mathbf{m}}(\sigma(i, i + 1)) = \mathcal{B}_i^{\mathbf{m}}(\sigma) \kappa^{\mathbf{m}}(\sigma) \quad (m_{\sigma(i)} > m_{\sigma(i+1)}). \tag{6.15}$$

Proof. We prove the lemma by induction in the length of elements in S^m . For $\sigma = \text{id}$ the (6.14) trivially holds with $\kappa^{\mathbf{m}}(\text{id}) = 1$. Fix a $\sigma \in S^m$ and assume that (6.14) is true for all elements of lengths less or equal to $l(\sigma) - 1$. Then by the property (5.6), $(i, i + 1)$ and $\tilde{\sigma} \in S^m$ exist such that $\sigma = \tilde{\sigma}(i, i + 1)$, $m_{\tilde{\sigma}(i)} > m_{\tilde{\sigma}(i+1)}$ and $l(\tilde{\sigma}) = l(\sigma) - 1$. By writing

$$K_{\sigma} \Phi^{\mathbf{m}}(z) = K_{i, i+1} K_{\tilde{\sigma}} \Phi^{\mathbf{m}}(z) \tag{6.16}$$

by the inductive assumption and (5.11) we obtain the desired statement. \square

Since the $\bar{\sigma}$ is the element of maximal length in S^m from this lemma and (6.13) we find

$$\psi_{\bar{\sigma}} = (-1)^{l(\bar{\sigma})} \left(\prod_{s=1}^M p_s! \right) \kappa^{\mathbf{m}}(\bar{\sigma}) P_{\bar{\sigma}} \omega_{\mathbf{m}} = (-1)^{l(\bar{\sigma})} \left(\prod_{s=1}^M p_s! \right) \kappa^{\mathbf{m}}(\bar{\sigma}) \bar{\omega}_{\mathbf{m}} \tag{6.17}$$

where

$$\bar{\omega}_{\mathbf{m}} = P_{\bar{\sigma}} \omega_{p_1} \otimes \omega_{p_2} \otimes \dots \otimes \omega_{p_M} = \omega_{p_M} \otimes \omega_{p_{M-1}} \otimes \dots \otimes \omega_{p_1}. \tag{6.18}$$

Now solving the recursion relation (6.15) with the initial condition $\kappa^{\mathbf{m}}(\text{id}) = 1$ we obtain

$$\kappa^{\mathbf{m}}(\bar{\sigma}) = \prod_{\substack{1 \leq i < j \leq N \\ m_i > m_j}} \frac{(\xi_i^{\mathbf{m}} - \xi_j^{\mathbf{m}})^2 - 1}{(\xi_i^{\mathbf{m}} - \xi_j^{\mathbf{m}})^2} = \prod_{1 \leq s < t \leq M} \frac{(h_m^{(s)} - h_m^{(t)} - p_s)(h_m^{(s)} - h_m^{(t)} + p_t)}{(h_m^{(s)} - h_m^{(t)})(h_m^{(s)} - h_m^{(t)} + p_t - p_s)}. \tag{6.19}$$

On the other hand, by using the recursion relation (5.27) we obtain

$$\check{\mathbb{R}}^{(-)}(\bar{\sigma})\omega_{\mathbf{m}} = (-1)^{l(\bar{\sigma})} \left(\prod_{1 \leq s < t \leq M} a_{s,t}(h_m^{(s)} - h_m^{(t)}) \right) \bar{\omega}_{\mathbf{m}} \tag{6.20}$$

where

$$a_{s,t}(x) := \begin{cases} \frac{x + p_t}{x + p_t - p_s} & (p_s \leq p_t) \\ \frac{x + p_t}{x} & (p_s \geq p_t). \end{cases} \tag{6.21}$$

Hence introducing

$$\rho(\mathbf{m}) := \left(\prod_{1 \leq s < t \leq M} \rho_{s,t}(h_m^{(s)} - h_m^{(t)}) \right) \tag{6.22}$$

$$\rho_{s,t}(x) := \begin{cases} \frac{x}{x - p_s} & (p_s \leq p_t) \\ \frac{x + p_t - p_s}{x - p_s} & (p_s \geq p_t). \end{cases} \tag{6.23}$$

We find from (6.19) and (6.21) that

$$c(\mathbf{m}) = \left(\prod_{s=1}^M p_s! \right) \frac{1}{\rho(\mathbf{m})} \tag{6.24}$$

and

$$\langle \Omega_m^{(-)}, \Omega_m^{(-)} \rangle_{(-)} = N! \rho(\mathbf{m}) \langle \Phi^m(z), \Phi^m(z) \rangle_c. \tag{6.25}$$

Proposition 11. For $\mathbf{m} \in \mathcal{M}_N$ we have $(\xi_i^m := \alpha m_i - i)$:

$$\langle \Phi^m(z), \Phi^m(z) \rangle_c = \frac{1}{N!} \prod_{1 \leq i < j \leq N} \frac{\Gamma\left(\frac{\xi_i^m - \xi_j^m}{\alpha} + \frac{1}{\alpha} + 1\right) \Gamma\left(\frac{\xi_i^m - \xi_j^m}{\alpha} - \frac{1}{\alpha} + 1\right)}{\left\{ \Gamma\left(\frac{\xi_i^m - \xi_j^m}{\alpha} + 1\right) \right\}^2} \tag{6.26}$$

or, equivalently, in notations (5.23), (5.24):

$$\begin{aligned} \langle \Phi^m(z), \Phi^m(z) \rangle_c &= \frac{1}{N!} \left(\prod_{s=1}^M \frac{\Gamma\left(\frac{p_s}{\alpha} + 1\right)}{\left\{ \Gamma\left(\frac{1}{\alpha} + 1\right) \right\}^{p_s}} \right) \\ &\times \prod_{1 \leq s < t \leq M} \frac{\Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} + \frac{p_t}{\alpha} + 1\right) \Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} - \frac{p_s}{\alpha} + 1\right)}{\Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} + \frac{p_t - p_s}{\alpha} + 1\right) \Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} + 1\right)}. \end{aligned} \tag{6.27}$$

Proof. To prove the proposition we will use the known formula for the norms of symmetric Jack polynomials. The Jack polynomial $P_m^{(\alpha)}(z)$ [19] is the unique symmetric vector in the space $E^m := \oplus_{\sigma \in S^m} \mathbb{C} \Phi_\sigma^m(z)$ ($\mathbf{m} \in \mathcal{M}_N$) normalized so that in the expansion

$$P_m^{(\alpha)}(z) = \sum_{\sigma \in S^m} v^m(\sigma) \Phi_\sigma^m(z) \quad (v^m(\sigma) \in \mathbb{R}) \tag{6.28}$$

the coefficient $v^m(\text{id})$ is equal to 1. The symmetry conditions

$$K_{i,i+1} P_m^{(\alpha)}(z) = P_m^{(\alpha)}(z) \quad (i = 1, 2, \dots, N - 1) \tag{6.29}$$

together with formulae (5.11), (5.12) give the recursion relation

$$\begin{aligned} v^m(\sigma(i, i + 1)) &= (1 - \mathcal{A}_i(\sigma)) v^m(\sigma) \\ &= \frac{\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma) - 1}{\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma)} v^m(\sigma) \quad (m_{\sigma(i)} > m_{\sigma(i+1)}). \end{aligned} \tag{6.30}$$

Solving this relation with the initial condition $v^m(\text{id}) = 1$ gives

$$v^m(\bar{\sigma}) = \prod_{\substack{1 \leq i < j \leq N \\ m_i > m_j}} \frac{\xi_i^m - \xi_j^m - 1}{\xi_i^m - \xi_j^m} \tag{6.31}$$

where $\bar{\sigma}$ is the element of maximal length in the set S^m (6.11).

Let $\text{Symm}_N := \sum_{\sigma \in \mathfrak{S}_N} K_\sigma$ be the symmetrization operator in $\mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_N^{\pm 1}]$. Then

$$\text{Symm}_N \Phi^m(z) = d(\mathbf{m}) P_m^{(\alpha)}(z) \quad (d(\mathbf{m}) \in \mathbb{R}). \tag{6.32}$$

Writing

$$\text{Symm}_N \Phi^m(z) = \sum_{\sigma \in S^m} K_\sigma \sum_{\tau \in \mathfrak{S}_N^m} K_\tau \Phi^m(z) \tag{6.33}$$

and using (5.13) and the result of lemma 2 we obtain

$$\text{Symm}_N \Phi^m(z) = \left(\prod_{s=1}^M p_s! \right) \kappa^m(\bar{\sigma}) \Phi_{\bar{\sigma}}^m(z) + \sum_{\substack{\sigma \in S^m \\ \sigma \neq \bar{\sigma}}} \zeta(\sigma) \Phi_\sigma^m(z) \quad (\zeta(\sigma) \in \mathbb{R}) \tag{6.34}$$

where $\kappa^m(\bar{\sigma})$ is given by (6.19). Comparing the last equation with (6.28) gives for the coefficient $d(\mathbf{m})$ in (6.32):

$$d(\mathbf{m}) = \left(\prod_{s=1}^M p_s! \right) \frac{\kappa^m(\bar{\sigma})}{v^m(\bar{\sigma})} = \prod_{1 \leq i < j \leq N} \frac{\xi_i^m - \xi_j^m + 1}{\xi_i^m - \xi_j^m}. \tag{6.35}$$

Now the self-adjointness of the symmetrization operator with respect to the scalar product (2.3) yields

$$\langle \Phi^m(z), \Phi^m(z) \rangle_c = \frac{1}{N!} d(\mathbf{m}) \langle P_m^{(\alpha)}(z), P_m^{(\alpha)}(z) \rangle_c. \tag{6.36}$$

Using the expression [19, ch VI-10.38]:

$$\langle P_m^{(\alpha)}(z), P_m^{(\alpha)}(z) \rangle_c = \prod_{1 \leq i < j \leq N} \frac{\Gamma\left(\frac{\xi_i^m - \xi_j^m}{\alpha} + \frac{1}{\alpha}\right) \Gamma\left(\frac{\xi_i^m - \xi_j^m}{\alpha} - \frac{1}{\alpha} + 1\right)}{\Gamma\left(\frac{\xi_i^m - \xi_j^m}{\alpha} + 1\right) \Gamma\left(\frac{\xi_i^m - \xi_j^m}{\alpha}\right)} \quad (\xi_i^m := \alpha m_i - i) \tag{6.37}$$

and (6.35) we obtain (6.26). Formula (6.27) follows from (6.26) by using the notations (5.23) and (5.24). \square

Now, by combining the result of proposition 11 and formula (6.25) we obtain the main result of this section.

Proposition 12. For $\mathbf{m} \in \mathcal{M}_N^{(n)}$ we have

$$\begin{aligned} \langle \Omega_m^{(-)}, \Omega_m^{(-)} \rangle_{(-)} &= \prod_{s=1}^M \frac{\Gamma\left(\frac{p_s}{\alpha} + 1\right)}{\left\{ \Gamma\left(\frac{1}{\alpha} + 1\right) \right\}^{p_s}} \\ &\times \prod_{1 \leq s < t \leq M} \frac{\Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} + \frac{p_t}{\alpha} + 1\right) \Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} - \frac{p_s}{\alpha}\right)}{\Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} + \frac{p_t - p_s}{\alpha} + \theta(p_s \leq p_t)\right) \Gamma\left(\frac{h_m^{(s)} - h_m^{(t)}}{\alpha} + \theta(p_s > p_t)\right)} \end{aligned} \tag{6.38}$$

where

$$\theta(x) := \begin{cases} 1 & \text{when } x \text{ is true} \\ 0 & \text{when } x \text{ is false.} \end{cases} \tag{6.39}$$

6.2. *The norm of the ground state in the fermionic case*

The ground state $\Omega_{\mathbf{m}^0(N)}^{(-)}$ of the fermionic SCSM is identified with the highest-weight vector in the Yangian subrepresentation $F^{m^0(N)}$ where the ground-state partition $\mathbf{m}^0(N)$ is described as follows. For a given number of particles, N , let $L \in \{0\} \cup \mathbb{N}$ and $q \in \{0, 1, \dots, n-1\}$ be defined by $N = nL + q$. Then

$$\mathbf{m}^0(N) = \begin{cases} (L)^n(L-1)^n \dots (1)^n(0)^q & (q \neq 0) \\ (L-1)^n(L-2)^n \dots (1)^n(0)^n & (q = 0) \end{cases} \tag{6.40}$$

where we used the usual convention: $(a)^r = \underbrace{a, a, \dots, a}_r$. The ground state $\Omega_{\mathbf{m}^0(N)}^{(-)}$ has

degeneracy equal to $\dim F^{m^0(N)} = \binom{n}{q}$.

One expression for $\Omega_{\mathbf{m}^0(N)}^{(-)}$ is given by formula (5.31). For the special case of the ground-state partition $\mathbf{m}^0(N)$ this expression can be simplified by taking into account the triangularity of the non-symmetric Jack polynomials (5.9). This gives

$$\Omega_{\mathbf{m}^0(N)}^{(-)} = \mathfrak{A}_N^{(-)}(z_1^{m^0(N)}, z_2^{m^0(N)} \dots z_N^{m^0(N)}) \otimes ((v_1 \otimes v_2 \otimes \dots \otimes v_n)^{\otimes L}) \otimes v_1 \otimes v_2 \otimes \dots \otimes v_q. \tag{6.41}$$

Let us introduce the Laurent polynomials $\tilde{f}_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N)$ and $f_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N)$ by

$$\begin{aligned} \tilde{f}_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N) &:= \prod_{\epsilon=1}^q \left(\prod_{(\epsilon-1)(L+1) < i < j \leq \epsilon(L+1)} (z_i - z_j) \right) \\ &\times \prod_{\epsilon=q+1}^n \left(\prod_{q+(\epsilon-1)L < i < j \leq q+\epsilon L} (z_i - z_j) \right) \end{aligned}$$

and

$$f_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N) := \begin{cases} \left(\prod_{i=q(L+1)+1}^N z_i \right) \tilde{f}_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N) & (q \neq 0) \\ \tilde{f}_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N) & (q = 0). \end{cases}$$

And let the sequence $(\epsilon_1^0, \epsilon_2^0, \dots, \epsilon_N^0)$ be defined as follows:

$$(\epsilon_1^0, \epsilon_2^0, \dots, \epsilon_N^0) := \begin{cases} (1)^{L+1}(2)^{L+1} \dots (q)^{L+1}(q+1)^L \dots (n)^L & (q \neq 0) \\ (1)^L(2)^L \dots (n)^L & (q = 0). \end{cases} \tag{6.42}$$

Then up to a sign the ground state $\Omega_{\mathbf{m}^0(N)}^{(-)}$ can be represented as

$$\sum_{\sigma} (-1)^{l(\sigma)} f_{\mathbf{m}^0(N)}(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(N)}) \otimes (v_{\epsilon_{\sigma(1)}^0} \otimes v_{\epsilon_{\sigma(2)}^0} \otimes \dots \otimes v_{\epsilon_{\sigma(N)}^0}) \tag{6.43}$$

where the sum is taken over all permutations such that the corresponding sequences $(\epsilon_{\sigma(1)}^0, \epsilon_{\sigma(2)}^0, \dots, \epsilon_{\sigma(N)}^0)$ are all distinct.

Using this presentation we can write the norm of the ground state as

$$\begin{aligned} \langle \Omega_{\mathbf{m}^0(N)}^{(-)}, \Omega_{\mathbf{m}^0(N)}^{(-)} \rangle_{(-)} &= \frac{N!}{\{(L+1)!\}^q \{L!\}^{n-q}} \\ &\times \langle f_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N), f_{\mathbf{m}^0(N)}(z_1, z_2, \dots, z_N) \rangle_{\mathcal{L}}. \end{aligned} \tag{6.44}$$

By definition (2.3) of the scalar product on the space of Laurent polynomials we can now recast the statement of proposition 12 for the case of the ground state as the following integral formula

$$\begin{aligned} \langle \Omega_{m^0(N)}^{(-)}, \Omega_{m^0(N)}^{(-)} \rangle_{(-)} &= \frac{1}{\{(L+1)!\}^q \{L!\}^{n-q}} \left(\prod_{i=1}^N \oint_{|w_i|=1} \frac{dw_i}{2\pi\sqrt{-1}w_i} \right) \\ &\times \prod_{i<j} |w_i - w_j|^{\frac{2}{\alpha}} |f_{m^0(N)}(w_1, w_2, \dots, w_N)|^2 = \frac{\Gamma\left(\left(\frac{n}{\alpha} + 1\right)L + \frac{q}{\alpha} + 1\right)}{L! \left(\frac{n}{\alpha} + 1\right)^L \Gamma\left(\frac{1}{\alpha} + 1\right)^N}. \end{aligned} \tag{6.45}$$

6.3. The bosonic case

The computation of the norms of the Yangian highest-weight vectors in the bosonic case is much simpler than that in the fermionic case. From equation (5.45) and the definition of the Jack polynomial we immediately find

$$\Omega_m^{(+)} = P_m^{(\alpha)}(z) \otimes (v_1^{\otimes N}) \quad (\mathbf{m} \in \mathcal{M}_N). \tag{6.46}$$

Hence the norm of the highest-weight vector is given by

$$\langle \Omega_m^{(+)}, \Omega_m^{(+)} \rangle_{(+)} = \langle P_m^{(\alpha)}(z), P_m^{(\alpha)}(z) \rangle_c \tag{6.47}$$

where the norm $\langle P_m^{(\alpha)}(z), P_m^{(\alpha)}(z) \rangle_c$ of the Jack polynomial is given by formula (6.37).

7. Eigenbases of the Gelfand–Zetlin algebra in the irreducible Yangian submodules and norms of the eigenvectors

In this section we construct eigenbases of the operator-valued series $A_1^{(\kappa)}(u)$, $A_2^{(\kappa)}(u), \dots, A_n^{(\kappa)}(u)$ within each of the irreducible $Y(\mathfrak{gl}_n)$ -submodules $F^{\mathbf{m}}$ ($\mathbf{m} \in \mathcal{M}_N^{(n)}$) ($\kappa = -1$ —fermionic case) and $B^{\mathbf{m}}$ ($\mathbf{m} \in \mathcal{M}_N$) ($\kappa = 1$ —bosonic case), and compute norms of the eigenvectors that form these eigenbases.

Due to the isomorphisms given by theorems 1 and 2 the construction of the eigenbases is carried out by a straightforward application of the results of Nazarov and Tarasov that are summarized in section 3.

Let us fix a partition $\mathbf{m} = (m_1, m_2, \dots, m_N) \in \mathcal{M}_N$ and let for $\kappa = -1$ $\mathbf{m} \in \mathcal{M}_N^{(n)} \subset \mathcal{M}_N$. As in section 5 associate with \mathbf{m} the following data:

M —the number of distinct elements in the sequence $\mathbf{m} = (m_1, m_2, \dots, m_N)$; p_s ($s = 1, 2, \dots, M$)—the multiplicities of the elements in the \mathbf{m} :

$$\begin{aligned} m_1 = m_2 = \dots = m_{p_1} > m_{1+p_1} = m_{2+p_1} = \dots = m_{p_2+p_1} > \dots > m_{1+p_{M-1}+\dots+p_2+p_1} \\ &= m_{2+p_{M-1}+\dots+p_2+p_1} = \dots = m_{p_M+\dots+p_2+p_1} \equiv N. \end{aligned} \tag{7.1}$$

Since in the fermionic case the partition is restricted: $\mathbf{m} \in \mathcal{M}_N^{(n)}$, we have $p_s \in \{1, 2, \dots, n\}$ ($s = 1, 2, \dots, M$) when $\kappa = -1$.

With $\xi_i^{\mathbf{m}} := \xi_i^{\mathbf{m}}(\text{id}) = \alpha m_i - i$ (5.10) set

$$h_m^{(s)} := -\kappa \xi_{1+p_1+p_2+\dots+p_{s-1}}^{\mathbf{m}} \quad (p_0 := 0, \quad s = 1, 2, \dots, M). \tag{7.2}$$

For $p \in \{1, 2, \dots, n\}$ let $S_p^{(-)}$ denote the set of all Gelfand–Zetlin schemes Λ that are associated with the irreducible \mathfrak{gl}_n -module with the highest weight (cf section 3)

$$\underbrace{(1, 1, \dots, 1)}_p, \underbrace{(0, 0, \dots, 0)}_{n-p}. \tag{7.3}$$

An element of $\mathcal{S}_p^{(-)}$ is an array of the form

$$\begin{array}{c} \lambda_{n,1}\lambda_{n,2}\cdots\cdots\cdots\lambda_{n,n} \\ \lambda_{n-1,1}\cdots\cdots\lambda_{n-1,n-1} \\ \vdots \\ \lambda_{2,1}\lambda_{2,2} \\ \lambda_{1,1} \end{array} \tag{7.4}$$

where

$$\begin{aligned} (\lambda_{m,1}, \lambda_{m,2}, \dots, \lambda_{m,m}) &= (\underbrace{1, 1, \dots, 1}_{l_m}, \underbrace{0, 0, \dots, 0}_{m-l_m}) \quad (m = 1, 2, \dots, n) \\ l_n &= p \end{aligned} \tag{7.5}$$

and either

$$l_m = l_{m+1} \quad \text{or} \quad l_m = l_{m+1} - 1 \quad (m = 1, 2, \dots, n - 1). \tag{7.6}$$

For $p \in \mathbb{N}$ let $\mathcal{S}_p^{(+)}$ denote the set of all Gelfand–Zetlin schemes $\Lambda = \{\lambda_{m,m'} / n \geq m \geq m' \geq 1\}$ that are associated with the irreducible \mathfrak{gl}_n -module with the highest weight (cf section 3)

$$(p, \underbrace{0, 0, \dots, 0}_{n-1}). \tag{7.7}$$

An element of $\mathcal{S}_p^{(+)}$ is a Gelfand–Zetlin scheme of the form

$$\begin{array}{c} \alpha_n 0 \cdots \cdots \cdots 0 \\ \alpha_{n-1} 0 \cdots \cdots \cdots 0 \\ \vdots \\ \alpha_2 0 \\ \alpha_1 \end{array} \tag{7.8}$$

where

$$\alpha_m \leq \alpha_{m+1} \quad (m = 1, 2, \dots, n - 1) \quad \alpha_n = p. \tag{7.9}$$

Now let us define the following operator-valued series.

For the bosonic case set

$$\begin{aligned} a_m^{(+)}(u) &= A_m^{(+)}(u) & b_m^{(+)}(u) &= B_m^{(+)}(u) \\ c_m^{(+)}(u) &= C_m^{(+)}(u) & d_m^{(+)}(u) &= D_m^{(+)}(u). \end{aligned} \tag{7.10}$$

And for the fermionic case set

$$\begin{aligned} a_m^{(-)}(u) &= \Delta(u)A_m^{(-)}(u) & b_m^{(-)}(u) &= \Delta(u)B_m^{(-)}(u) \\ c_m^{(-)}(u) &= \Delta(u)C_m^{(-)}(u) & d_m^{(-)}(u) &= \Delta(u)D_m^{(-)}(u) \end{aligned} \tag{7.11}$$

where $\Delta(u) = \prod_{i=1}^N (u + d_i)$. Then from proposition 10 it follows that

$$a_m^{(\kappa)}(u)^\dagger = a_m^{(\kappa)}(u) \quad b_m^{(\kappa)}(u)^\dagger = c_m^{(\kappa)}(u) \quad c_m^{(\kappa)}(u)^\dagger = b_m^{(\kappa)}(u) \quad \kappa = -, +. \tag{7.12}$$

For a collection of Gelfand–Zetlin schemes $\Lambda^{(1)}, \dots, \Lambda^{(M)}$ such that $\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(\kappa)}$ ($s = 1, 2, \dots, M$) define the following vector (cf section 3):

$$v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} = \prod_{(m,m')}^{\rightarrow} \left(\prod_{\substack{(s,t) \\ 1 \leq t \leq \lambda_{n,m'}^{(s)} - \lambda_{m,m'}^{(s)}}} b_m^{(\kappa)}(v_{m,m'}^{(s)} - t) \right) \Omega_m^{(\kappa)} \tag{7.13}$$

$$v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} \in \begin{cases} F^m & (\kappa = -) \\ B^m & (\kappa = +). \end{cases} \tag{7.14}$$

Here

$$v_{m,m'}^{(s)} = m' - \lambda_{m,m'}^{(s)} - 1 - h_m^{(s)} \tag{7.15}$$

and the $h_m^{(s)}$ are defined by (7.2). From proposition 8 and theorems 1 and 2 it follows that the set

$$\{v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} | \Lambda^{(s)} \in \mathcal{S}_{p_s}^{(\kappa)} \quad (s = 1, 2, \dots, M)\} \tag{7.16}$$

is a base of F^m (resp. B^m) when $\kappa = -$ (resp. $+$). Due to proposition 4 this is an eigenbase of the operators generating the Gelfand–Zetlin algebra:

$$A_m^{(\kappa)}(u) v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} = \mathcal{A}_m^{(\kappa)}(u; \mathbf{m})_{\Lambda^{(1)}, \dots, \Lambda^{(M)}} v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} \quad (m = 1, 2, \dots, n) \tag{7.17}$$

where the eigenvalues are defined by

$$\mathcal{A}_m^{(-)}(u; \mathbf{m})_{\Lambda^{(1)}, \dots, \Lambda^{(M)}} = \prod_{s=1}^M \frac{u + 1 + h_m^{(s)}}{u + 1 + h_m^{(s)} - l_m^{(s)}} \quad (\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(-)}) \tag{7.18}$$

$$\mathcal{A}_m^{(+)}(u; \mathbf{m})_{\Lambda^{(1)}, \dots, \Lambda^{(M)}} = \prod_{s=1}^M \frac{u + h_m^{(s)} + \alpha_m^{(s)}}{u + h_m^{(s)}} \quad (\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(+)}) \tag{7.19}$$

Since $\langle \Phi_\sigma^m(z), \Phi_\tau^n(z) \rangle_c = 0$ when $m \neq n$, the subspaces F^m (resp. B^m) are pairwise orthogonal.

For $\alpha > 0$ one can verify, that the data $\mathbf{m} \in \mathcal{M}_N$, $(\Lambda^{(1)}, \Lambda^{(2)}, \dots, \Lambda^{(M)}) (\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(\kappa)})$ are uniquely restored from the collection of rational functions

$$\mathcal{A}_1^{(\kappa)}(u; \mathbf{m})_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}, \mathcal{A}_2^{(\kappa)}(u; \mathbf{m})_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}, \dots, \mathcal{A}_n^{(\kappa)}(u; \mathbf{m})_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}. \tag{7.20}$$

That is the joint spectrum of eigenvalues of the Gelfand–Zetlin algebra is simple. Since $A_m^{(\kappa)}(u)$ are self-adjoint, we obtain the following.

Proposition 13. For $\mathbf{m} \in \mathcal{M}_N^{(n)}$ (resp. $\mathbf{m} \in \mathcal{M}_N$) the set

$$\{v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} | \Lambda^{(s)} \in \mathcal{S}_{p_s}^{(\kappa)} (s = 1, 2, \dots, M)\} \tag{7.21}$$

with $\kappa = -$ (resp. $\kappa = +$) is an orthogonal base of F^m (resp. B^m).

The norms of the eigenvectors $v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)}$ are as follows.

Proposition 14. Bosonic case. Let $\mathbf{m} \in \mathcal{M}_N$ and $\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(+)} (s = 1, 2, \dots, M)$. If we write a Gelfand–Zetlin scheme $\Lambda^{(s)}$ as in (7.8):

$$\begin{array}{c} \Lambda^{(s)} = \alpha_n^{(s)} 0 \dots \dots \dots 0 \\ \alpha_{n-1}^{(s)} 0 \dots \dots \dots 0 \\ \vdots \dots \dots \\ \alpha_2^{(s)} 0 \\ \alpha_1^{(s)} \end{array} \tag{7.22}$$

then the norm of the vector $v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(+)}$ is

$$\begin{aligned} \langle v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(+)}, v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(+)} \rangle_{(+)} &= \langle \Omega_{\mathbf{m}}^{(+)}, \Omega_{\mathbf{m}}^{(+)} \rangle_{(+)} \\ &\times \prod_{1 \leq m \leq n} \left\{ \prod_{1 \leq s \leq M} \frac{(\alpha_n^{(s)} - \alpha_m^{(s)})! (\alpha_n^{(s)} - \alpha_{m-1}^{(s)})! (\alpha_m^{(s)})^2}{(\alpha_m^{(s)} - \alpha_{m-1}^{(s)})! (\alpha_n^{(s)})^2} \right\} \\ &\times \left\{ \prod_{\substack{(s,s') \\ s \neq s'}} \prod_{a=\alpha_m^{(s)}}^{\alpha_n^{(s)}-1} \frac{(-a + \alpha_n^{(s')} + h_m^{(s')} - h_m^{(s)}) (-1 - a + \alpha_{m-1}^{(s')} + h_m^{(s')} - h_m^{(s)})}{(-1 - a + h_m^{(s')} - h_m^{(s)})^2} \right\} \end{aligned}$$

$$\times \prod_{\substack{(s,s') \\ s < s'}} \frac{(\alpha_n^{(s')} - \alpha_n^{(s)} + h_m^{(s')} - h_m^{(s)})}{(\alpha_m^{(s')} - \alpha_m^{(s)} + h_m^{(s')} - h_m^{(s)})}$$

where $h_m^{(s)}$ are defined by (7.2) with $\kappa = +$.

Fermionic case. Let $\mathbf{m} \in \mathcal{M}_N^{(n)}$ and $\Lambda^{(s)} \in \mathcal{S}_{p_s}^{(-)} (s = 1, 2, \dots, M)$. As in (7.5) define $l_m^{(s)}$ associated with the Gelfand–Zetlin scheme $\Lambda^{(s)}$ by the conditions $\lambda_{m,l_m^{(s)}}^{(s)} = 1$ and $\lambda_{m,l_m^{(s)}+1}^{(s)} = 0$. Then the norm of the vector $v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(-)}$ is

$$\begin{aligned} \langle v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(-)}, v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(-)} \rangle_{(-)} &= \langle \Omega_{\mathbf{m}}^{(-)}, \Omega_{\mathbf{m}}^{(-)} \rangle_{(-)} \left\{ \prod_{1 \leq s \leq M} \prod_{\substack{(m,m') \\ \lambda_{m,m'}^{(s)} \neq \lambda_{n,m'}^{(s)}}} (m' - 1)!^2 (p_s + 1 - m')!^2 \right\} \\ &\times \left\{ \prod_{\substack{(s,s') \\ s < s'}} \prod_{\substack{(m,m') \\ \lambda_{m,m'}^{(s)} \neq \lambda_{n,m'}^{(s)} \\ \lambda_{m,m'}^{(s')} \neq \lambda_{n,m'}^{(s')}}} \left[\prod_{j=0}^{p_s} (m' - j - 1 + h_m^{(s')} - h_m^{(s)})^2 \right. \right. \\ &\times \left. \left. \prod_{j=0}^{p_{s'}} (m' - j - 1 + h_m^{(s)} - h_m^{(s')})^2 \right] [(h_m^{(s)} - h_m^{(s')})^4]^{-1} \right\} \\ &\times \left\{ \prod_{\substack{(s,s') \\ s \neq s'}} \prod_{\substack{(m,m') \\ \lambda_{m,m'}^{(s)} \neq \lambda_{n,m'}^{(s)} \\ \lambda_{m,m'}^{(s')} = \lambda_{n,m'}^{(s')}}} \left[(m' - l_m^{(s')} + h_m^{(s')} - h_m^{(s)}) \prod_{j=0}^{p_s} (m' - j - 1 + h_m^{(s')} - h_m^{(s)})^2 \right] \right. \\ &\times [(m' - 1 - l_{m-1}^{(s')} + h_m^{(s')} - h_m^{(s)}) (m' - 1 - l_m^{(s')} + h_m^{(s')} - h_m^{(s)}) \\ &\times (m' - l_{m+1}^{(s')} + h_m^{(s')} - h_m^{(s)})]^{-1} \left. \right\} \end{aligned}$$

where $h_m^{(s)}$ are defined by (7.2) with $\kappa = -$. In these product formulae s and s' range from 1 to M (7.1) and (m, m') ($n \geq m \geq m' \geq 1$) are coordinates of points in a Gelfand–Zetlin scheme of \mathfrak{gl}_n . We give the proof in appendix B.

Remark. If $\alpha > 0$ we can confirm directly that the norms of the previous proposition are positive. The key points are as follows. For the bosonic case, if $s < s'$ then $h^{(s')} - h^{(s)} > p_s, \alpha_k^{(s)} \leq p_s$. For the fermionic case, if $s < s'$ then $h^{(s')} - h^{(s)} < -p_s, 1 \leq m' \leq p_s, l_k^{(s)} \leq p_s$.

Together with proposition 12 and formula (6.47) this proposition gives the norm formulae for the orthogonal eigenbasis of the SCSM.

8. Concluding remarks

In this paper we have constructed an orthogonal basis of eigenvectors for the SCSM and have derived product formulae for their norms. Our construction is based on the Gelfand–Zetlin algebra associated with the Yangian symmetry of the model. It is now natural to ask: what other properties of the eigenvectors are described in this paper? What we have in mind is exemplified by the scalar case, where the orthogonal eigenvectors are described by the symmetric Jack polynomials. For the Jack polynomials a number of properties such as triangularity, Cauchy formulae, duality, existence of associated symmetric functions etc

are known [19, 25]. We believe that most of these properties have their counterparts for the Calogero–Sutherland model with spin, we plan to report on this subject in the future.

Appendix A. Proof of theorem 1

Recall that for $\mathbf{m} \in \mathcal{M}_N$ the subspace $W_{(-)}^{\mathbf{m}} \subset \otimes^N \mathbb{C}^n$ was defined in (5.18) as follows:

$$W_{(-)}^{\mathbf{m}} := \bigcap_{i:m_i=m_{i+1}} \text{Ker}(P_{i,i+1} + 1) \tag{A.1}$$

and that from this definition it follows, in particular, that the dimension of the $W^{\mathbf{m}}$ is zero unless $\mathbf{m} \in \mathcal{M}_N^{(n)}$ where the set $\mathcal{M}_N^{(n)}$ is defined in (5.19).

Proposition 15. If $f \in W_{(-)}^{\mathbf{m}}$ then $U_{(-)}^{\mathbf{m}}f \in F^{\mathbf{m}}$, and the map $U_{(-)}^{\mathbf{m}} : f \rightarrow U_{(-)}^{\mathbf{m}}f$ is an isomorphism of the linear spaces $W_{(-)}^{\mathbf{m}}$ and $F^{\mathbf{m}}$.

Proof. For an arbitrary $\psi \in E^{\mathbf{m}} \otimes (\otimes^N \mathbb{C}^n)$ we write

$$\psi = \sum_{\sigma \in S^{\mathbf{m}}} \Phi_{\sigma}^{\mathbf{m}}(z) \otimes \psi_{\sigma} \tag{A.2}$$

where the components $\psi_{\sigma} \in \otimes^N \mathbb{C}^n$ are uniquely determined by ψ . We have $\psi \in F^{\mathbf{m}}$ if and only if

$$K_{i,i+1}\psi = -P_{i,i+1}\psi \quad (i = 1, 2, \dots, N - 1). \tag{A.3}$$

By virtue of (5.11), (5.12) and the linear independence of $\Phi_{\sigma}^{\mathbf{m}}(z)$ equations (A.3) are equivalent to

$$(P_{i,i+1} + 1)\psi_{\sigma} = 0 (m_{\sigma(i)} = m_{\sigma(i+1)}) \quad (\text{for all } \sigma \in S^{\mathbf{m}} \text{ and } i = 1, 2, \dots, N - 1) \tag{A.4}$$

$$\psi_{\sigma(i,i+1)} = \begin{cases} -\check{R}_{i,i+1}(x)\psi_{\sigma} & (m_{\sigma(i)} > m_{\sigma(i+1)}) \\ -\frac{x^2}{x^2 - 1}\check{R}_{i,i+1}(x)\psi_{\sigma} & (m_{\sigma(i)} < m_{\sigma(i+1)}) \end{cases} \quad x := \xi_i^{\mathbf{m}}(\sigma) - \xi_{i+1}^{\mathbf{m}}(\sigma). \tag{A.5}$$

Notice that the second equation (the case $m_{\sigma(i)} < m_{\sigma(i+1)}$) in (A.5) is not independent but follows from the first one (the case $m_{\sigma(i)} > m_{\sigma(i+1)}$).

For any $\sigma \in S^{\mathbf{m}}$ define a set L_{σ} whose elements are sets $\{\psi_{\tau}\}_{\tau \in S^{\mathbf{m}}} (\psi_{\tau} \in \otimes^N \mathbb{C}^n)$. We will say that $\{\psi_{\tau}\}_{\tau \in S^{\mathbf{m}}} \in L_{\sigma}$ if and only if the following relations are satisfied

$$(P_{i,i+1} + 1)\psi_{\sigma} = 0 \quad (\text{for all } i \text{ s.t. } m_{\sigma(i)} = m_{\sigma(i+1)}) \tag{A.6}$$

and for all $\tau \in S^{\mathbf{m}}$ and $i = 1, 2, \dots, N - 1$

$$\psi_{\tau(i,i+1)} = \begin{cases} -\check{R}_{i,i+1}(x)\psi_{\tau} & (m_{\tau(i)} > m_{\tau(i+1)}) \\ -\frac{x^2}{x^2 - 1}\check{R}_{i,i+1}(x)\psi_{\tau} & (m_{\tau(i)} < m_{\tau(i+1)}) \end{cases} \quad x := \xi_i^{\mathbf{m}}(\tau) - \xi_{i+1}^{\mathbf{m}}(\tau). \tag{A.7}$$

With this definition we have

$$\psi \in F^{\mathbf{m}} \Leftrightarrow (\text{A.4}), (\text{A.5}) \Leftrightarrow \{\psi_{\sigma}\}_{\sigma \in S^{\mathbf{m}}} \in \bigcap_{\sigma \in S^{\mathbf{m}}} L_{\sigma}. \tag{A.8}$$

Lemma 3.

$$\bigcap_{\sigma \in S^{\mathbf{m}}} L_{\sigma} = L_{\text{id}}. \tag{A.9}$$

Proof. We will prove that for any $\sigma \in S^m, \sigma \prec \text{id}$ the inclusion

$$\{\psi_\tau\}_{\tau \in S^m} \in \bigcap_{\sigma' \succ \sigma} L_{\sigma'} \tag{A.10}$$

implies

$$\{\psi_\tau\}_{\tau \in S^m} \in L_\sigma. \tag{A.11}$$

Then, since id is the maximal element in S^m , induction of the order of S^m will give

$$L_{\text{id}} \subset \bigcap_{\sigma \in S^m} L_\sigma \tag{A.12}$$

and the statement of the lemma will follow.

Fix a $\sigma \in S^m, \sigma \prec \text{id}$ and assume that (A.10) holds. For any $\sigma \in S^m, \sigma \prec \text{id}, i \in \{1, 2, \dots, N - 1\}$ exists such that $m_{\sigma(i)} < m_{\sigma(i+1)}$ (otherwise σ must be equal to id).

With this i let $\sigma' := \sigma(i, i + 1)$. Then $\sigma' \in S^m$, and by the definition of the ordering in S^m (5.7) we have $\sigma' \succ \sigma$.

Now take any $j \in \{1, 2, \dots, N - 1\}$ such that $m_{\sigma(j)} = m_{\sigma(j+1)}$. If such a j does not exist, the implication (A.10) \Rightarrow (A.11) is obvious.

The following three situations may take place:

$$(1) \quad |j - i| \geq 2 \tag{A.13}$$

$$(2) \quad j = i + 1 \tag{A.14}$$

$$(3) \quad j = i - 1. \tag{A.15}$$

If (1) holds, then $m_{\sigma'(j)} = m_{\sigma'(j+1)}$, and by the assumption

$$\{\psi_\tau\}_{\tau \in S^m} \in \bigcap_{\sigma' \succ \sigma} L_{\sigma'} \tag{A.16}$$

we have

$$(P_{j,j+1} + 1)\psi_{\sigma'} = 0. \tag{A.17}$$

Relations (A.7) give

$$\psi_{\sigma'} = \bar{R}_{i,i+1}(x)\psi_\sigma \quad \left(\bar{R}_{i,i+1}(x) := -\frac{x^2}{x^2 - 1} \check{R}_{i,i+1}(x), x := \xi_i^m(\sigma) - \xi_{i+1}^m(\sigma) \right). \tag{A.18}$$

And hence

$$(P_{j,j+1} + 1)\psi_\sigma = 0 \tag{A.19}$$

because $x := \xi_i^m(\sigma) - \xi_{i+1}^m(\sigma) = \alpha(m_{\sigma(i)} - m_{\sigma(i+1)}) + \sigma(i + 1) - \sigma(i) < -1$ when $\alpha > 0$ and $m_{\sigma(i)} < m_{\sigma(i+1)}$, and therefore the $\bar{R}_{i,i+1}(x)$ is invertible.

Now let situation (2) hold (that is $j = i + 1$). Then

$$m_{\sigma(i)} < m_{\sigma(i+1)} = m_{\sigma(i+2)} \tag{A.20}$$

$$\sigma(i + 2) = \sigma(i + 1) + 1. \tag{A.21}$$

Let $\sigma'' := \sigma'(i, i + 1) = \sigma(i, i + 1)(i + 1, i + 2)$. We have

$$\begin{aligned} \sigma''(i) &= \sigma'(i) = \sigma(i + 1) \\ \sigma''(i + 1) &= \sigma'(i + 2) = \sigma(i + 2) \\ \sigma''(i + 2) &= \sigma'(i + 1) = \sigma(i) \end{aligned} \tag{A.22}$$

and hence, by (A.20) $\sigma'' \succ \sigma' \succ \sigma$.

By (A.7) and (A.22) one has

$$\psi_{\sigma''} = \bar{R}_{i+1,i+2}(\xi_i^m(\sigma) - \xi_{i+2}^m(\sigma)) \bar{R}_{i,i+1}(\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma)) \psi_{\sigma}. \tag{A.23}$$

By assumption (A.16) we have

$$(P_{i,i+1} + 1)\psi_{\sigma''} = 0. \tag{A.24}$$

Since $\xi_{i+1}^m(\sigma) - \xi_{i+2}^m(\sigma) = 1$, by the Yang–Baxter equation

$$\begin{aligned} &\check{R}_{i,i+1}(\xi_{i+1}^m(\sigma) - \xi_{i+2}^m(\sigma)) \check{R}_{i+1,i+2}(\xi_i^m(\sigma) - \xi_{i+2}^m(\sigma)) \check{R}_{i,i+1}(\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma)) \\ &= \check{R}_{i+1,i+2}(\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma)) \check{R}_{i,i+1}(\xi_i^m(\sigma) - \xi_{i+2}^m(\sigma)) \\ &\quad - \xi_{i+2}^m(\sigma) \check{R}_{i+1,i+2}(\xi_{i+1}^m(\sigma) - \xi_{i+2}^m(\sigma)) \end{aligned}$$

and by $\check{R}_{i,i+1}(1) = P_{i,i+1} + 1$ we obtain from (A.23), (A.24)

$$\bar{R}_{i+1,i+2}(\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma)) \bar{R}_{i,i+1}(\xi_i^m(\sigma) - \xi_{i+2}^m(\sigma)) (P_{i+1,i+2} + 1)\psi_{\sigma} = 0. \tag{A.25}$$

Now $(P_{i+1,i+2} + 1)\psi_{\sigma} = 0$ follows by the invertibility of the operators $\bar{R}_{i+1,i+2}(\xi_i^m(\sigma) - \xi_{i+1}^m(\sigma))$ and $\bar{R}_{i,i+1}(\xi_i^m(\sigma) - \xi_{i+2}^m(\sigma))$.

Situation (3) is considered in virtually the same way as (2) to show that (A.16) entails

$$(P_{i-1,i} + 1)\psi_{\sigma} = 0. \tag{A.26}$$

Thus (A.10) implies (A.11) and the lemma is proven. \square

From this lemma and (A.8) we obtain

$$\psi \in F^m \Leftrightarrow \{\psi_{\sigma}\}_{\sigma \in S^m} \in L_{\text{id}}. \tag{A.27}$$

Now we are in a position to show that $v \in W_{(-)}^m$ implies $U_{(-)}^m v \in F^m$. Indeed, by the definitions of $\check{\mathbb{R}}^{(-)}(\sigma)$ (5.27) and $W_{(-)}^m$ (5.18) we have $\{\check{\mathbb{R}}^{(-)}(\sigma)v\}_{\sigma \in S^m} \in L_{\text{id}}$ and hence $U_{(-)}^m v = \sum_{\sigma \in S^m} \Phi_{\sigma}^m(z) \otimes \check{\mathbb{R}}^{(-)}(\sigma)v$ belongs to F^m as implied by (A.27).

Next, we demonstrate surjectivity of the map $U_{(-)}^m : W_{(-)}^m \rightarrow F^m$. Let

$$\psi = \sum_{\sigma \in S^m} \Phi_{\sigma}^m(z) \otimes \psi_{\sigma} \in F^m. \tag{A.28}$$

Then $\{\psi_{\sigma}\}_{\sigma \in S^m} \in L_{\text{id}} \Rightarrow \psi_{\text{id}} \in W_{(-)}^m$ and by solving relations (A.7) we find

$$\psi_{\sigma} = \check{\mathbb{R}}^{(-)}(\sigma)\psi_{\text{id}} \quad (\sigma \in S^m). \tag{A.29}$$

Hence $\psi = U_{(-)}^m \psi_{\text{id}}$ and the surjectivity follows.

Now suppose $U_{(-)}^m v = 0$, $v \in W_{(-)}^m$. Due to the linear independence of the non-symmetric Jack polynomials $\Phi_{\sigma}^m(z)$ we obtain

$$\check{\mathbb{R}}^{(-)}(\sigma)v = 0 \quad (\sigma \in S^m) \tag{A.30}$$

and in particular $v = 0$ which shows injectivity of the map $U_{(-)}^m$. This completes the proof of the proposition. \square

For commuting operators (or complex numbers) a_i ($i = 1, 2, \dots, N$) let $T_0(u)$ be the following monodromy operator

$$T_0(u, \{a_i\}) := L_{0,1}(u, a_1)L_{0,2}(u, a_2) \cdots L_{0,N}(u, a_N). \tag{A.31}$$

Here the L -operator is

$$L_{0,i}(u, a_i) := 1 + \frac{P_{0,i}}{u + a_i}. \tag{A.32}$$

The subspace F^m for any $m \in \mathcal{M}_N^{(n)}$ is a $Y(\mathfrak{gl}_n)$ -module with the action given by the monodromy operator (4.1), (4.4) $\hat{T}_0(u) = T_0(u, \{d_i\})$.

The space $W_{(-)}^m$ (5.18) is also a $Y(\mathfrak{gl}_n)$ -module. Now the action is specified by the monodromy operator $T_0(u, \{\xi_i^m(\text{id})\})$. Indeed with this action we have the identity of the Yangian modules (5.25):

$$W_{(-)}^m = V_{p_1}(h_m^{(1)}) \otimes V_{p_2}(h_m^{(2)}) \otimes \cdots \otimes V_{p_M}(h_m^{(M)}) \tag{A.33}$$

which is established by the standard fusion procedure taking into account that $\xi_{i+1}^m(\text{id}) = \xi_i^m(\text{id}) - 1$ whenever $m_i = m_{i+1}$.

Proposition 16. The map $U_{(-)}^m : W_{(-)}^m \longrightarrow F^m$ is an intertwiner of the $Y(\mathfrak{gl}_n)$ -modules.

Proof. The intertwining property of the R -matrix:

$$\begin{aligned} L_{0,i}(u, \xi_i^m(\sigma))L_{0,i+1}(u, \xi_{i+1}^m(\sigma))\check{R}_{i,i+1}(\xi_i^m(\sigma(i, i+1)) - \xi_{i+1}^m(\sigma(i, i+1))) \\ = \check{R}_{i,i+1}(\xi_i^m(\sigma(i, i+1)) - \xi_{i+1}^m(\sigma(i, i+1))) \\ \times L_{0,i}(u, \xi_i^m(\sigma(i, i+1)))L_{0,i+1}(u, \xi_{i+1}^m(\sigma(i, i+1))) \end{aligned} \tag{A.34}$$

and (5.10) entail the following chain of equations ($v \in W_{(-)}^m$):

$$\begin{aligned} \hat{T}_0(u)U_{(-)}^m v &= \sum_{\sigma \in S^m} \Phi_{\sigma}^m(z) \otimes T_0(u; \{\xi_i^m(\sigma)\})\check{R}^{(-)}(\sigma)v \\ &= \sum_{\sigma \in S^m} \Phi_{\sigma}^m(z) \otimes \check{R}^{(-)}(\sigma)T_0(u, \{\xi_i^m(\text{id})\})v = U_{(-)}^m T_0(u, \{\xi_i^m(\text{id})\})v. \end{aligned} \tag{A.35}$$

□

Propositions 15 and 16 imply the statement of theorem 1.

Appendix B. Proof of proposition 14

Let us define the vector

$$\bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} = U_{(\kappa)}^m \left(\prod_{\substack{(s,t) \\ 1 \leq t \leq \lambda_{n,m'}^{(s)} - \lambda_{m,m'}^{(s)}}} b_m(v_{m,m'}^{(s)} - t) \right) (U_{(\kappa)}^m)^{-1} \Omega_m^{(\kappa)} \quad (\kappa = -, +) \tag{B.1}$$

where $b_m(u)$ and $v_{m,m'}^{(s)} - t$ are defined in (3.37), (7.15). Notice that the following relations are satisfied:

$$v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} = f_{\kappa}(\Lambda^{(1)}, \dots, \Lambda^{(M)}) \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)} \quad \text{for some scalar function } f_{\kappa}(\cdot). \tag{B.2}$$

The calculation of the function $f_{\kappa}(\cdot)$ can be done by comparing the ratio of $b_m(u)$ with $b_m^{(\kappa)}(u)$ ($\kappa = -, +$). We will calculate the norms of $\bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)}$. Then we will obtain the norms of $v_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)}$.

To calculate the norms of $\bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(M)}}^{(\kappa)}$, we will derive recursion relations between

$$(\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}) \quad \text{and} \quad (\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m,m'}, \dots, \Lambda^{(M)})$$

and will solve them. Here $\Lambda^{(i)} + e_{m,m'}$ is the Gelfand–Zetlin scheme, whose (j, j') -elements are $\lambda_{j,j'} + \delta_{j,m} \delta_{j',m'}$.

Proposition 17.

$$\begin{aligned} \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)}, \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} &= \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} \\ &\times \varpi_{m+1, +}(\nu) \varpi_{m-1, +}(\nu - 1) \varpi_{m, +}(\nu)^{-1} \bar{\varpi}_{m, +}(\nu - 1). \end{aligned} \tag{B.3}$$

Here $\nu = m' - \lambda_{m, m'}^{(s)} - 1 - h_m^{(s)}$, and $\varpi_{k, +}(u)$, $\bar{\varpi}_{k, +}(u)$ are defined by the following relations:

$$\varpi_{k, +}(u) \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} = a_k(u) \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \tag{B.4}$$

$$\bar{\varpi}_{k, +}(u) = \lim_{u' \rightarrow u} (u - u') \varpi_{k, +}(u'). \tag{B.5}$$

Proof. By using relation (7.12) and (B.2), we obtain

$$\begin{aligned} \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)}, \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} &= \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m \\ &\times c_m(\nu - 1) b_m(\nu - 1) (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} \\ &= \lim_{\nu' \rightarrow \nu} \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m c_m(\nu - 1) b_m(\nu' - 1) \\ &\times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)}. \end{aligned} \tag{B.6}$$

On the other hand, relation (3.42) gives

$$\begin{aligned} c_m(\nu - 1) b_m(\nu' - 1) &= b_m(\nu' - 1) c_m(\nu - 1) \\ &+ \frac{1}{\nu - \nu'} \{d_m(\nu - 1) a_m(\nu' - 1) - d_m(\nu' - 1) a_m(\nu - 1)\}. \end{aligned} \tag{B.7}$$

Since $a_m(\nu - 1) (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} = 0$, and

$$\lim_{\nu' \rightarrow \nu} \frac{1}{\nu - \nu'} a_m(\nu' - 1) \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} = \bar{\varpi}_{m, +}(\nu - 1) \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}$$

we have

$$\begin{aligned} \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)}, \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} &= \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m b_m(\nu - 1) c_m(\nu - 1) \\ &\times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} + \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m d_m(\nu - 1) \\ &\times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} \bar{\varpi}_{m, +}(\nu - 1) \\ &= \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m b_m(\nu - 1) c_m(\nu - 1) \\ &\times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} + \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m b_m(\nu - 1) \\ &\times c_m(\nu) (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} \bar{\varpi}_{m, +}(\nu - 1) \varpi_{m, +}(\nu)^{-1} \\ &+ \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} \bar{\varpi}_{m, +}(\nu - 1) \varpi_{m, +}(\nu)^{-1} \\ &\times \varpi_{m+1, +}(\nu) \varpi_{m-1, +}(\nu - 1). \end{aligned} \tag{B.8}$$

In (B.8), we used relation (3.43). Then if we show the following lemma, we have proven proposition 17. \square

Lemma 4. In the situation of proposition 17, we have

$$\begin{aligned} \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m b_m(\nu - 1) c_m(\nu - 1) (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} \\ + \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m b_m(\nu - 1) c_m(\nu) \\ \times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m, m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} \bar{\varpi}_{m, +}(\nu - 1) \varpi_{m, +}(\nu)^{-1} = 0. \end{aligned} \tag{B.9}$$

Proof. By using relation (3.42), we can show inductively that if $\Lambda^{(i)} + te_{m,m'} \in \mathcal{S}_{\lambda^{(r)}}$ then

$$\begin{aligned} \text{l.h.s. of (B.9)} &= \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m b_m(v-1) \cdots b_m(v-t) c_m(v-1) \\ &\quad \times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + te_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)} + U_{(\kappa)}^m b_m(v-1) \cdots b_m(v-t) c_m(v) \\ &\quad \times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + te_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \bar{\omega}_{m,+}(v-1) \bar{\omega}_{m,+}(v)^{-1} \rangle_{(\kappa)}. \end{aligned} \tag{B.10}$$

Let t be the maximal number st $\Lambda^{(i)} + te_{m,m'} \in \mathcal{S}_{\lambda^{(r)}}$. From relations (7.12), we obtain

$$\begin{aligned} \text{l.h.s. of (B.9)} &= \langle U_{(\kappa)}^m c_m(v-t) \cdots c_m(v-1) (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)}, U_{(\kappa)}^m c_m(v-1) \\ &\quad \times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + te_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)} + U_{(\kappa)}^m c_m(v) \\ &\quad \times (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + te_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)} \bar{\omega}_{m,+}(v-1) \bar{\omega}_{m,+}(v)^{-1} \rangle_{(\kappa)}. \end{aligned} \tag{B.11}$$

If we apply repeatedly theorem 3.5 from [23] (here $\gamma_{m,m'}^{(i)}$ is some constant),

$$\begin{aligned} U_{(\kappa)}^m c_m(v_{m,m'}^{(i)}) (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)} &= \begin{cases} \gamma_{m,m'}^{(i)} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)} & \text{if } \Lambda^{(i)} + e_{m,m'} \in \mathcal{S}_{\lambda^{(r)}} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{B.12}$$

we have

$$U_{(\kappa)}^m c_m(v-t) \cdots c_m(v-1) (U_{(\kappa)}^m)^{-1} \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)} + e_{m,m'}, \dots, \Lambda^{(M)}}^{(\kappa)} = 0. \tag{B.13}$$

So we obtain lemma 4. □

Let $\backslash \kappa_{m,m'} /$ be the Gelfand–Zetlin scheme which corresponds to the highest-weight vector, the highest weights are $(\lambda_{n,1}, \dots, \lambda_{n,n})$ (i.e. $\kappa_{m,m'} = \lambda_{n,m'}$ for all possible m, m'). If we solve the recursive relations of proposition 17, we obtain the following.

Proposition 18.

$$\begin{aligned} \langle \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)}, \bar{v}_{\Lambda^{(1)}, \dots, \Lambda^{(i)}, \dots, \Lambda^{(M)}}^{(\kappa)} \rangle_{(\kappa)} &= \langle \bar{v}_{\text{hwv}}^{(\kappa)}, \bar{v}_{\text{hwv}}^{(\kappa)} \rangle_{(\kappa)} \prod_{\substack{(m,m') \\ m' \leq m}} \left\{ \prod_{\substack{(s,s') \\ \text{for all pairs}}} \left\{ \prod_{a=\lambda_{m,m'}^{(s)}}^{\kappa_{m,m'}^{(s)} - 1} \left\{ (-1)^{\delta_{s,s'}} \right. \right. \right. \\ &\quad \times \prod_{j=1}^{m'} (m' - j - a + \kappa_{m+1,j}^{(s')} + h_m^{(s')} - h_m^{(s)}) \\ &\quad \times \prod_{j=m'+1}^{m+1} (m' - j - a + \lambda_{m+1,j}^{(s')} + h_m^{(s')} - h_m^{(s)}) \\ &\quad \times \prod_{j=1}^{m'-1} (m' - j - 1 - a + \kappa_{m-1,j}^{(s')} + h_m^{(s')} - h_m^{(s)}) \\ &\quad \left. \left. \left. \times \prod_{j=m'}^{m-1} (m' - j - 1 - a + \lambda_{m-1,j}^{(s')} + h_m^{(s')} - h_m^{(s)}) \right\} \right\} \right. \\ &\quad \left. \times \prod_{j=1}^{m'-1} \frac{(m' - j - \kappa_{m,m'}^{(s)} + \kappa_{m,j}^{(s')} + h_m^{(s')} - h_m^{(s)})}{(m' - j - \lambda_{m,m'}^{(s)} + \kappa_{m,j}^{(s')} + h_m^{(s')} - h_m^{(s)})} \right\} \end{aligned}$$

$$\begin{aligned} & \times \prod_{j=m'+1}^m \frac{(m' - j - \kappa_{m,m'}^{(s)} + \lambda_{m,j}^{(s')} + h_m^{(s')} - h_m^{(s)})}{(m' - j - \lambda_{m,m'}^{(s)} + \lambda_{m,j}^{(s')} + h_m^{(s')} - h_m^{(s)})} \Big\} \\ & \times \prod_{\substack{(s,s') \\ s < s'}} \frac{(-\kappa_{m,m'}^{(s)} + \kappa_{m,m'}^{(s')} + h_m^{(s')} - h_m^{(s)})}{(-\lambda_{m,m'}^{(s)} + \lambda_{m,m'}^{(s')} + h_m^{(s')} - h_m^{(s)})} \Big\}. \end{aligned}$$

If we rewrite proposition 18 for the bosonic (resp. the fermionic) case and take into account the function $f_\kappa(\cdot)$, we obtain proposition 14.

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